Def: (X,d) espaço métrico.

(1) $(x_n)_n \subseteq X$, $x_n \to x \in X$ se $\forall \varepsilon > 0 \; \exists N \in \mathbb{N}$ tal que

$$d(x_n, x) < \varepsilon, \; \forall n \geq N.$$

(2) $(x_n)_n$ é de Cauchy se $\forall \varepsilon > 0 \; \exists N \in \mathbb{N}$ tal que $\forall m, n \geq N$ vale:

$$d(x_m, x_n) < \varepsilon.$$

(3) (X,d) é completo se toda seq. de Cauchy converge para algum ponto de X.

Exemplos:

$I \subset \mathbb{R}^k$

a) $B(I) = \{ f : I \to \mathbb{R}^n \mid f$ é contínua e limitada $\}$.

$$d(f,g) = \sup_{x \in I} \| f(x) - g(x) \| < +\infty$$

Af: $(B(I), d)$ é completo.

$(f_n)_n$; $f_n \in B(I)$, $(f_n)_n$ Cauchy.

<u>Af:</u> $\exists f \in B(I)$ e $f_n \xrightarrow{d} f$.

$\forall x \in I$, $\quad \| f_n(x) - f_m(x) \| \leq d(f_n, f_m)$

$\Rightarrow \left(f_n(x) \right)_n$ é Cauchy em $\mathbb{R}^n$.

$\Rightarrow \exists\, v_x \in \mathbb{R}^n$ tq $\quad f_n(x) \xrightarrow{n} v_x$

$$\left[\begin{array}{l} f : I \longrightarrow \mathbb{R}^n \\ \quad x \longmapsto f(x) = v_x \end{array} \right]$$ candidato à limite

(I) $\quad f \in B(I)$

· f é limitada: $x \in I$.

$$\| f(x) \| = \| f(x) - f_{n_0}(x) + f_{n_0}(x) \|$$

$$\leq \underbrace{\| f(x) - f_{n_0}(x) \|}_{< \varepsilon} + \underbrace{\| f_{n_0}(x) \|}_{< M_{n_0}}$$

$f_n(x) \to f$

$n_0 \gg 0 \Rightarrow \| f(x) - f_{n_0}(x) \| < \varepsilon$.

$\Rightarrow \| f(x) \| < \varepsilon + M$, $\quad \| f(x) \| \leqslant M$

$\exists M$ tq se $n_0 \gg 0$ então

$$\|f_{n_0}(x)\| < M \quad , \forall n_0 \gg 0$$

$\Rightarrow \quad \|f(x)\| < M \quad , \forall x \in I \Rightarrow f$ é limitada.

II) fixamos $x_0 \in I$ e tome $\varepsilon > 0$. Quero $\delta > 0$ tal que:

$$\|f(x_0) - f(x)\| < \varepsilon \quad , \quad \forall x \in B(x_0, \delta).$$

Se $n \gg 0$, $\quad \|f(x) - f_n(x)\| < \varepsilon/3 \quad , \forall x \in I.$

$$\|f(x) - f(x_0)\| \leq \underbrace{\|f(x_0) - f_n(x_0)\|}_{< \varepsilon} + \underbrace{\|f_n(x_0) - f_n(x)\|}_{< \varepsilon, \text{ se } x \in B(x_0, \delta)} + \underbrace{\|f_n(x) - f(x)\|}_{< \varepsilon}$$

$\Rightarrow \quad \|f(x) - f(x_0)\| < 3\varepsilon \quad , \quad \forall x \in B(x_0, \delta)$

$\Rightarrow \quad f \in B(I).$

Agora $f_n \xrightarrow{d} f$. $\forall \varepsilon > 0$ $\exists N \in \mathbb{N}$ tal

que: $d(f_n, f) < \varepsilon$, $\forall n \geq N$.

$$\| f_n(x) - f(x) \| < \varepsilon, \quad \text{ } n \gg 0$$

$$\Rightarrow \quad d(f_n, f) = \sup_x \| f_n(x) - f(x) \| < \varepsilon.$$

$$\Rightarrow \quad d(f_n, f) \xrightarrow[n \to \infty]{} 0.$$

$I \subset \mathbb{R}^k$

b) $\mathcal{L}_L = \{ f : I \to \mathbb{R}^n \mid f$ é L-Lipschitz e limitada $\}$ $\subset \mathcal{B}(I)$

$(f_n)_n \subseteq \mathcal{L}_L$ de Cauchy em $\mathcal{L}_L$

$\Rightarrow (f_n)_n$ é Cauchy em $B(I)$

$\Rightarrow \exists f$ t.q. $d(f_n, f) \to 0$
$\in B(I)$

$\forall \varepsilon > 0$ existe $N \in \mathbb{N}$ tal que $\| f_n(x) - f(x) \| < \varepsilon$, $\forall x \in I$

$n \geq N$.

Fixe: $n_0 > N$.

$$\|f(x) - f(y)\| \leq \|f(x) - f_{n_0}(x)\| + \|f_{n_0}(x) - f_{n_0}(y)\| + \|f_{n_0}(y) - f(y)\|$$

$f_n \xrightarrow{d} f$

$$\leq \varepsilon + L\cdot\|x-y\| + \varepsilon = 2\varepsilon + L\|x-y\|$$

$$\|f(x) - f(y)\| \leq 2\varepsilon + L\|x-y\|.$$

Se $\boxed{A \leq \varepsilon + B}$, $\forall \varepsilon > 0 \Rightarrow A \leq B$.

Senão, i.e., se $\underbrace{A > B}$ $\Rightarrow$ $A - B > 0$.

$\varepsilon = \frac{A-B}{2}$

$$A \leq \frac{A-B}{2} + B \iff 2A \leq A + B$$

$$\Rightarrow \boxed{A \leq B}$$

$$\Rightarrow \|f(x) - f(y)\| \leq L\cdot\|x-y\|, \forall x,y \in I$$

$f \in \mathcal{L}_L$.

Contrações: $T: X \to Y$ t.q. $\exists \lambda \in (0,1)$ t.q.

$$d_Y(Tx, Ty) \le \lambda \cdot d_X(x,y), \quad \forall x,y \in X$$

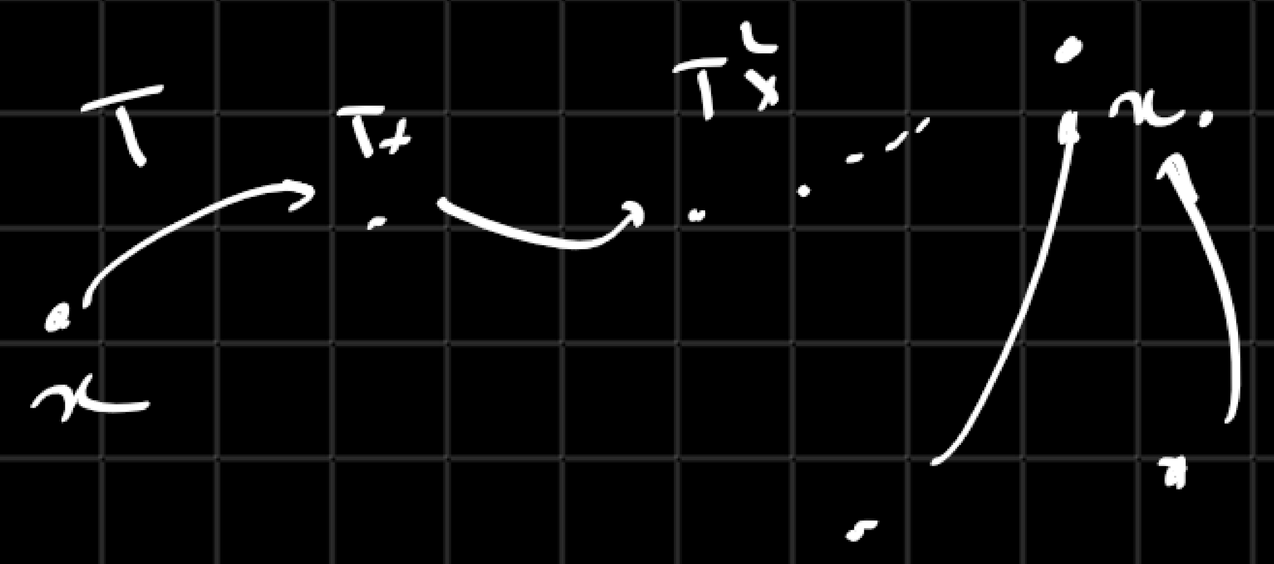
Teorema (Banach) Se (X,d) é completo e $T: X \to X$ é contração, então $\exists!\, x_0 \in X$ tq $T(x_0) = x_0$ e mais ainda, $\forall x \in X$, $T^n(x) \to x_0$.

x_0 é atrator global.

dem: Fixe $x \in X$. $T^n(x) \to x_0$.

$$d(T^n(x), x_0) \xrightarrow{n \to \infty} 0$$

$$\left[(x_n)_n = (T^n(x))_n \right]$$



Af: $(x_n)_n$ é Cauchy.

$m > n$

$$d(x_m, x_n) = d(T^m(x), T^n(x)) =$$

$$\le d(T^m(x), T^{m-1}(x)) + d(T^{m-1}(x), T^n(x)) \le d(T^m(x), T^{m-1}(x)) + d(T^{m-1}(x), T^{m-2}(x)) +$$
$$+ d(T^{m-2}(x), T^n(x)) \le \circledast$$

$$\circledast \le \sum_{i=n}^{m-1} d(T^{i+1}, T^i)$$

$m-1 \geq n$

$$\leq \left(\lambda^{m-1} + \lambda^{m-2} + \dots + \lambda^{n+1} + \lambda^{n} \right) d(Tx, x)$$

$$= \lambda^{n} \left(\lambda^{m-1-n} + \dots + 1 \right) d(Tx, x)$$

$$\left[(1 + \dots + \lambda^{n}) = \frac{1 - \lambda^{n+1}}{1 - \lambda}, \text{ se } \lambda \neq 1 \right]$$

$$= \lambda^{n} \left(\frac{1 - \lambda^{m-n}}{1 - \lambda} \right) d(Tx, x)$$

$m > n$

$$\leq \frac{\lambda^{n}}{1-\lambda} d(Tx, x) \xrightarrow{n \to \infty} 0 \qquad d(x_m, x_n) \xrightarrow{n \to \infty} 0$$

$\Rightarrow (x_n)_n$ é Cauchy.

X completo

$$\Rightarrow \exists\, x_0 \in X \text{ tq } \lim_{n \to \infty} T^{n}(x) = x_0$$

<u>Af:</u> x_0 é fixo. $\qquad T^{n} = T(T^{n-1})$

$$x_0 = \lim_{n} T^{n}(x) = T\big(\underbrace{\lim_{n} T^{n-1}(x)}_{x_0}\big)$$

contração $\Rightarrow$ contínua

Logo, $T(x_0) = x_0$.

<u>Af:</u> x_0 é único. $\qquad \exists\, y_0 \in X$ | y_0 fixo

$$d(x_0, y_0) = d(Tx_0, Ty_0) \leq \lambda\, d(x_0, y_0)$$

$$\underbrace{(1-\lambda)}_{>0} \cdot d(x_0, y_0) \leq 0 \Rightarrow d(x_0, y_0) = 0 \Rightarrow x_0 = y_0.$$

$\boxed{\text{Exercícios 1.12 e 1.13}}$: $A \in \mathcal{M}_n$.

$$\|A\| = \sup_{x \neq 0} \frac{\|Ax\|}{\|x\|}$$

<u>Af</u> $\|\cdot\|$ é norma.

- $\|A\| = \sup\{c\}$, $c \geq 0 \Rightarrow \|A\| \geq 0$

- $A = 0 \Rightarrow \|A\| = 0$; $\|A\| = 0$, $\displaystyle\sup_{x \neq 0} \frac{\|Ax\|}{\|x\|} = 0$

$$\Rightarrow \frac{\|Ax\|}{\|x\|} = 0 \quad \forall x \neq 0 \quad \Rightarrow \|Ax\| = 0, \ \forall x \neq 0$$

$$\Rightarrow Ax = 0 \ \forall x \neq 0, x = 0 \Rightarrow A = 0$$

- $\lambda \in \mathbb{R}$. $\displaystyle \|\lambda A\| = \sup_{x \neq 0} \frac{\|(\lambda A)x\|}{\|x\|} =$

$$= \sup_{x \neq 0} \frac{\|\lambda . (Ax)\|}{\|x\|} = \sup_{x \neq 0} |\lambda| . \frac{\|Ax\|}{\|x\|}$$

$$= |\lambda| \sup_{x \neq 0} \frac{\|Ax\|}{\|x\|} = |\lambda| . \|A\|$$

- $A, B \in \mathcal{M}_n$, $\displaystyle \|A + B\| = \sup \frac{\|(A+B)x\|}{\|x\|} = \sup\left(\frac{\|Ax + Bx\|}{\|x\|} \right.$

$$\leq \sup \left(\frac{\|Ax\|}{\|x\|} + \frac{\|Bx\|}{\|x\|} \right) \leq \underbrace{\sup \frac{\|Ax\|}{\|x\|}}_{\|A\|} + \underbrace{\sup \frac{\|Bx\|}{\|x\|}}_{\|B\|}$$

$\|A\| < +\infty$? $\|A\| = \sup_{x \neq 0} \frac{\|Ax\|}{\|x\|} = \sup_{\|x\|=1} \|Ax\|$. $A \in C^0$ + S compacto $\Rightarrow \|A\| < +\infty$.

$$d(Ax, Ay) = \|Ax - Ay\| = \|A(x-y)\| \leq \|A\| . \|x-y\| = \|A\| . d(x,y)$$

Se $\|A\| < 1 \Rightarrow A$ é contração.

$$A = \text{diag}(\lambda_1, \dots, \lambda_n)$$

$$\|A\| = \sup_{x \neq 0} \frac{\|Ax\|}{\|x\|} = \sup \frac{\|(\lambda_1 x_1, \dots, \lambda_n x_n)\|}{\|x\|} \leq \sup \frac{|\lambda| . \cancel{\|x\|}}{\cancel{\|x\|}} = |\lambda|$$

$$\|x\| = \left[\sum (x_i)^2 \right]^{1/2}$$

$\|(\lambda_1 x_1, \dots, \lambda_n x_n)\| \leq |\lambda| . \|x\|$,

$\lambda = \max_i |\lambda_i|$.

Se $|\lambda_i| < 1 \quad \forall i = 1, \dots, n. \Rightarrow A$ é contração

$$\sup_{x\neq 0} \frac{\|ABx\|}{\|x\|} \qquad x \neq 0.$$

$$\|ABx\| = \|A(\underbrace{Bx}_{v})\| \leq \|A\|.\|v\| = \|A\|.\underline{\|Bx\|}$$
$$\leq \|A\|.\|B\|.\|x\|$$

$$\forall x\neq 0 \qquad \frac{\|ABx\|}{\|x\|} \leq \underbrace{\|A\|.\|B\|}_{:=\lambda\, \in \mathbb{R}}$$

$$\Rightarrow \quad \sup_{x\neq 0} \frac{\|ABx\|}{\|x\|} \leq \lambda$$

$$\|AB\| \leq \|A\|.\|B\| \qquad ///$$

$\boxed{\text{Exercício } 1.11}$ (X,d) é esp. completo e $T: X \to X$ tal que $\exists\, m \geq 1$ com $T^m: X \to X$ é contração.

Então T possui um único ponto fixo atrator global.

Como T^m é contr., existe $p \in X$ ponto fixo atrator global: $T^m(p) = p$.

$$(T^m)^k(x) \to p.$$

$$x \xrightarrow{T^m} T^m(x) \quad T^{2m}(x) \quad T^{3m}(x) \longrightarrow p. \qquad T^m(x')\to p$$

$$(Tx) \quad (T^{m+1}(x)) \quad T^{2m+1}(x) \longrightarrow p$$

$$T^2(x) \quad T^{2+m}(x) \quad T^{2+2m}(x) \to p$$

$$\vdots$$

$$T^{m-1}(x) \quad T^{2m-1}(x) \quad \cdots \longrightarrow p$$

$$\mathbb{N} = \mathbb{N}_0 \cup \cdots \cup \mathbb{N}_{m-1}$$

$$\mathbb{N}_j = \{ j, j+m, j+2m, \ldots \}$$

•) $T^n(x) \longrightarrow p$ pois: $\forall \varepsilon > 0 \ \exists N \in \mathbb{N}$ tq $n \geq N \Rightarrow$

$d(T^n(x), p) < \varepsilon$ (basta tomar $N = \max\{n_0, \ldots, n_{m-1}\}$).

•) se p e q são ptos fixos $d(q,p) = d(T^m_q, T^m_p) \leq \lambda\, d(p,q)$

$$\underset{\lambda \in (0,1)}{\Longrightarrow} \quad p = q$$

•) Vamos mostrar que p é pto fixo de T:

$T^{m+1}(p) = T(T^m(p)) = T(p) \Rightarrow T^m(T(p)) = T(p)$

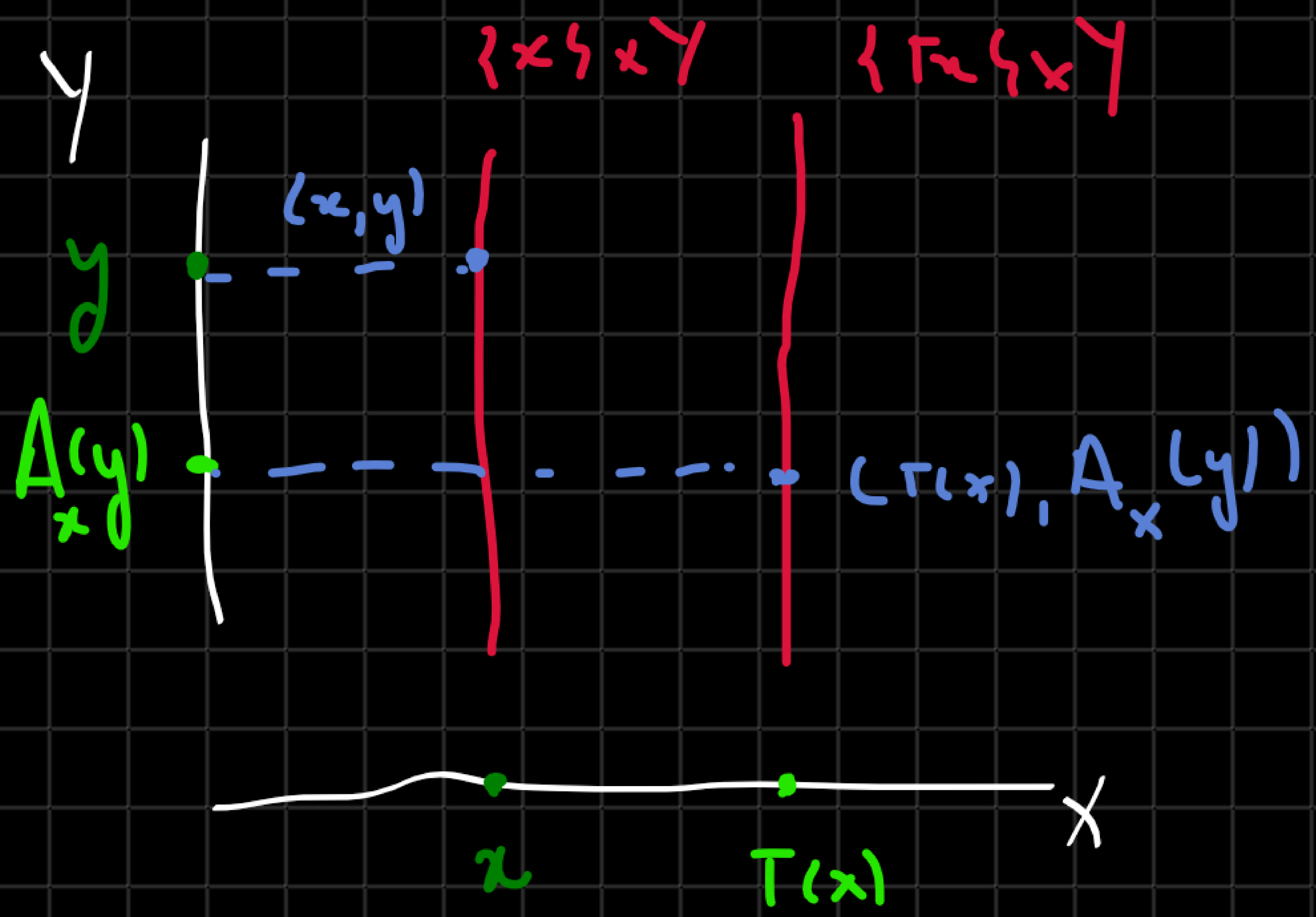
$\Rightarrow T(p)$ é pto fixo de $T^m \Rightarrow T(p) = p$. ///

$T: X \to X$

Def: (Skew product) $A: X \times Y \to Y$

$$S: X \times Y \to X \times Y$$

$$S(x,y) = (T(x), A(x,y))$$



$A_x : Y \to Y$

$y \mapsto A(x,y) = A_x(y)$

Contração nas fibras: se existe $\lambda \in (0,1)$ tal que

$$d_Y(A_x(y), A_x(\bar{y})) \leq \lambda \cdot d_Y(y, \bar{y}),$$

$\forall x \in X$, $\forall y, \bar{y} \in Y$.

Teorema das contrações em fibras:

$S: X \times Y \xrightarrow{C^0} X \times Y$, que contração nas fibras. Se $x_0 \in X$ é pto fixo atrator de T e y_0 é pto fixo de $A_{x_0}(y)$, então:

$\Rightarrow \quad (x_0, y_0) \in X \times Y$ é pont. fixo atrator de S.

dem: Seja $d((x,y),(\bar{x},\bar{y})) = d_X(x,\bar{x}) + d_Y(y,\bar{y})$

Vamos mostrar que:

$$d(S^n(x,y),(x_0,y_0)) \xrightarrow[n]{} 0$$

Como:

$$d(S^n(x,y),(x_0,y_0)) \leq d(S^n(x,y), S^n(x,y_0)) + d(S^n(x,y_0),(x_0,y_0))$$

vamos ver que cada parcela converge à zero.

Note que:

$$S^2(x,y) = S(S(x,y)) = S(Tx, A(x,y))$$

$$= (T(Tx), A(Tx, A(x,y)))$$

$$= (T^2x, A_{Tx}(A_x(y)))$$

$$= (T^2x, A_{Tx} \circ A_x(y))$$

$$\Rightarrow S^n(x,y) = (T^n(x), \underbrace{A_{T^{n-1}x} \circ \dots \circ A_{Tx} \circ A_x(y)}_{A_{n,x}(y)})$$

(I) $d(S^n(x,y), S^n(x,y_0)) =$

$$= d\big((T^n_x, A_{n,x}(y)), (T^n_x, A_{n,x}(y_0))\big)$$

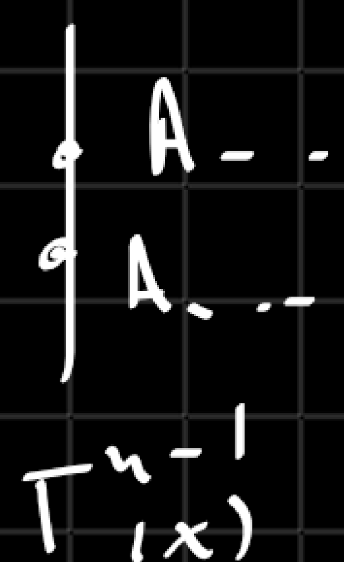
$$= d_Y\big(A_{n,x}(y), A_{n,x}(y_0)\big)$$

$$\le \lambda^n d(y,y_0) \xrightarrow[n\to\infty]{} 0$$

(II) $d(S^n(x,y_0), (x_0,y_0)) = d\big((T^n(x), A_{n,x}(y_0)), (x_0,y_0)\big)$

$$= \underbrace{d_X(T^n(x), x_0)}_{\xrightarrow[n\to\infty]{} 0} + \underbrace{d_Y(A_{n,x}(y_0), y_0)}_{?}$$

(III) $d_Y(A_{n,x}(y_0), y_0) \le$



$$\le \sum_{i=0}^{n-1} d_Y\big(A_{T^{n-1}(x)} \circ \dots \circ A_{T^i(x)}(y_0),$$

$$A_{T^{n-1}(x)} \circ \dots \circ A_{T^{i+1}(x)}(y_0)\big)$$

$$\le \sum_{i=0}^{n-1} \lambda^{n-i-1} \cdot d\big(A_{T^i(x)}(y_0), y_0\big)$$

Como $A_{x_0}(y_0) = y_0 = A_{T^i(x_0)}(y_0)$, definimos

$$c_i := d_Y\left(A_{T^i(x)}(y_0), \underbrace{A_{T^i(x_0)}(y_0)}_{=x_0}\right)$$ e obtemos:

$$\boxed{c_i \to 0 \quad , i \to \infty}.$$

De fato, como A é contínua, se: $(x_n, y_n) \to (x, y)$
então $A(x_n, y_n) \to A(x, y)$. Logo, se

$$(T^i(x), y_0) \to (x_0, y_0),$$

como $A(T^i(x), y_0)$, então $c_i \xrightarrow{i \to \infty} 0$.

Ainda, $c_i \to 0 \Rightarrow 0 \leq c_i < c$ $\exists N \in \mathbb{N}$ tq $0 \leq c_i < \varepsilon$.

Portanto a soma fica:

observe! → $$\sum_{i=1}^{n} \lambda^{n-i} \cdot c_i = \sum_{i=1}^{k-1} \lambda^{n-i} c_i + \sum_{i=k}^{n} \lambda^{n-i} c_i$$

$$\leq c \cdot \sum_{i=1}^{k} \lambda^{n-i} + \left(\sup_{j \geq k} c_j\right) \sum_{i=k}^{n} \lambda^{n-i}.$$

$$= c\left(\lambda^{n-1} + \lambda^{n-2} + \dots + \lambda^{k}\right) + \left(\sup_{j\geq k} c_j\right)\left(\lambda^{k-1} + \dots + 1\right)$$

$$\leq c \cdot \frac{\lambda^{n-k+1}}{1-\lambda} + \left(\sup_{j\geq k} c_j\right)\frac{1}{1-\lambda}.$$

Mas:

$$\limsup_{n\to\infty} \sum_{i=1}^{n} \lambda^{n-i} c_i \leq \left(\cancel{\sup_{j\geq k} c_j}^{\,0}\right) \frac{1}{1-\lambda} \xrightarrow[k\to\infty]{} 0$$

(quando $n\to\infty$ podemos fazer $k\to+\infty$)

Isso mostra que:

$$d_Y\left(A_{x,n}(y_0), y_0\right) \to 0$$

Juntando (I), (II) e (III), concluímos:

$$d\left(S^n(x,y), (x_0,y_0)\right) \to 0.$$

de sorte que (x_0,y_0) é um ponto fixo atrator. ///