Exercícios do Cap 1 (Barreira-Valls)

1.14 $Z \subset C(I)$, $I \subset \mathbb{R}^k$, I limitado

$$\bar{d}(x,y) = \sup\left\{ \frac{\|x(t)-y(t)\|}{\|t\|} ; t \in I \setminus \{0\} \right\}$$

$$\bar{d} : Z \times Z \to \mathbb{R}$$

Perg: $\bar{d}(x,y) = +\infty$ ou de fato, $\bar{d}(x,y) < \infty$?

- $\bar{d}(x,x) = 0$

$$\sup\left\{ \frac{\|x(t)-x(t)\|}{\|t\|} ; t \in I \setminus \{0\} \right\}$$

$$= \sup\{0\} = 0 \Rightarrow \bar{d}(x,x) = 0$$

- $\bar{d}(x,y) = 0 \Rightarrow x = y$

$$0 \leq \frac{\|x(t) - y(t)\|}{\|t\|} \leq 0 \ , \ \forall t \in I \setminus \{0\}$$

$$\Rightarrow \|x(t) - y(t)\| = 0 \ \forall t \in I \setminus \{0\}.$$

$$\Rightarrow x(t) = y(t) \quad \forall t \in I \setminus \{0\}$$

$$x(0) = y(0) = 0 \quad \Rightarrow \quad x \equiv y.$$

- $\bar{d}(x,y) = \bar{d}(y,x)$ (de graça)

- $\bar{d}(x,y) \leq \bar{d}(x,z) + \bar{d}(z,y)$

- Se $\forall a \in A$, $\exists b_a \in B$ tq
$$a \leq b_a \Rightarrow \sup A \leq \underbrace{\sup B}_{\beta}.$$

dem

$$a \leq b_a \leq \beta$$

$$\Rightarrow a \leq \beta, \ \forall a \in A \Rightarrow \sup A \leq \beta$$
$$\Rightarrow \sup A \leq \sup B.$$

- $\sup(A+B) \leq \underbrace{\sup A}_{\alpha} + \underbrace{\sup B}_{\beta}$.

$$a+b \leq \alpha + b \leq \alpha + \beta, \quad \forall a \in A, \ \forall b \in B$$

$\Rightarrow \forall a+b \in A+B$,

$$a+b \leq \alpha + \beta$$

$$\sup A+B \leq \alpha + \beta = \sup A + \sup B.$$

$$\bar{d}(x,y) = \sup\left\{ \frac{\|x(t) - y(t)\|}{\|t\|} ; t \in I \setminus \{0\} \right\} \leq$$

$$\leq \sup \underbrace{\left\{ \frac{\|x(t) - z(t)\|}{\|t\|} + \frac{\|z(t) - y(t)\|}{\|t\|} \right\}}_{A+B}$$

$$\leq \sup \underbrace{\left\{ \frac{\|x(t) - z(t)\|}{\|t\|} \right\}}_{A} + \sup \underbrace{\left\{ \frac{\|z(t) - y(t)\|}{\|t\|} \right\}}_{B}$$

$$= \bar{d}(x,z) + \bar{d}(z,y).$$

Portanto, $(Z, \bar{d})$ é espaço métrico.

Vamos ver que $(Z, \bar{d})$ é completo.

Seja $(x_n)_n$ seq. de Cauchy em Z.

$\forall \varepsilon > 0 \; \exists N \geq 1$ tal que

$$\bar{d}(x_m, x_n) < \varepsilon,$$

$\forall m, n > N$.

Quem seria $x \in Z$ tq $x_n \xrightarrow{\bar{d}} x$?

Para cada $t \in I$, observe:

$$\bar{d}(x_m, x_n) < \varepsilon, \quad m, n \geq N.$$

$$\shortparallel$$

$$\sup \left\{ \frac{\|x_m(t) - x_n(t)\|}{\|t\|} \right\}$$

$$\vee\!\!/$$

$$\|x_m(t_0) - x_n(t_0)\| < \varepsilon . \|t_0\| < \varepsilon . |I|$$

$|I| = \operatorname{diam} I < +\infty$

$\forall t_0$ fixado, $(x_n(t_0))_n$ é Cauchy:

$$\| x_m(t_0) - x_n(t_0) \| < \varepsilon . |I| ,$$

$\forall m,n \geq N$.

$(x_n(t_0))_n \subseteq \mathbb{R}^n \Rightarrow x_n(t_0) \to x(t_0)$

Fica definida $x : I \to \mathbb{R}^n$.
$t \mapsto x(t)$

Quero mostrar: $x_n \xrightarrow{\bar{d}} x$ e

$x \in Z$.

- $x_n \xrightarrow{\bar{d}} x$?

- $x \in Z$. Se $x_n \xrightarrow{\bar{d}} x$, então ok!

$$| x(t_1) - x(t_2) | \leq | x(t_1) - x_n(t_1) | +$$
$$+ | x_n(t_1) - x_n(t_2) | + | x_n(t_2) - x(t_2) | \leq$$
$$\leq \varepsilon . |t_1| + L . |t_1 - t_2| + |t_2| . \varepsilon \leq 2 . |I| . \varepsilon + L . |t_1 - t_2|$$

$$|x(t_1) - x(t_2)| \leq 2.|I|.\varepsilon + L|t_1 - t_2|$$

fazendo $\varepsilon \to 0$,

$$|x(t_1) - x(t_2)| \leq L.|t_1 - t_2|$$

$\Rightarrow$ x é L-Lipschitz.

x é limitada pois

$n >> 0$.

$$\bar{d}(x_n, x) < \varepsilon$$

$$\Rightarrow \sup \left\{ \frac{|x_n(t) - x(t)|}{|t|} \right\} < \varepsilon$$

$$\Rightarrow \quad \underset{\displaystyle |x(t)| - |x_n(t)|}{\underset{\rotatebox{90}{$\leq$}}{|x_n(t) - x(t)|}} < \varepsilon.|t| \leq \varepsilon.|I|.$$

$$|x(t)| \leq \varepsilon.|I| + |x_n(t)| \leq \varepsilon.|I| + c_{x_n}, \quad \forall t \in I.$$

$\Rightarrow$ x é limitada

1.15

$$x' = f(x) \qquad (1)\begin{cases} x' = f(x) \\ x(t_0) = x_0 \end{cases}$$

$$f : \Omega \to \mathbb{R}^n$$

Posso definir $\varphi(t, t_0, x_0) = \tilde{\varphi}(t, 0, x_0)$,

ou seja não perco informação.

De fato, se $\gamma : I_{t_0} \to \Omega$ é solução

de (1) então: $\tilde{\gamma} : I_0 \to \Omega$ é solução

de $(2)\begin{cases} x' = f(x) \\ x(0) = x_0 \end{cases}$, $I_0 = I_{t_0} - t_0$

$$\tilde{\gamma}(t)' = \gamma'(t+t_0) \cdot 1 = f(\gamma(t+t_0)),$$

$$\forall t \in I_0.$$

e

$$\tilde{\gamma}(0) = \gamma(0+t_0) = \gamma(t_0) = x_0.$$

Buscamos uma expressão para $f(x_0)$.

$$\frac{\partial}{\partial t} \underbrace{\varphi(t, 0, x_0)}_{\frac{\partial}{\partial t}\gamma(t)} = f(\underbrace{\varphi(t, 0, x_0)}_{\gamma(t)})$$

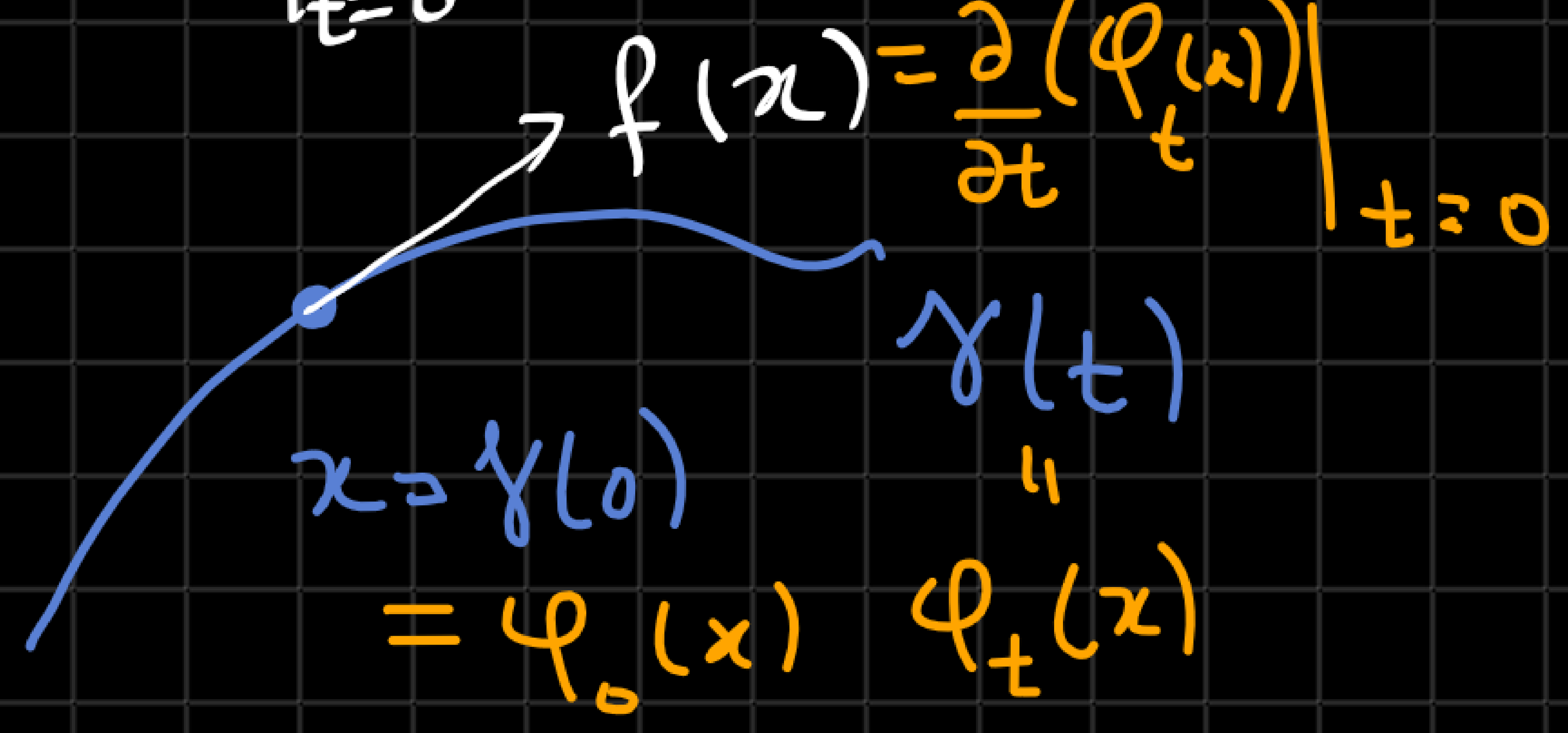
Se $t = 0 \Rightarrow \varphi(t, 0, x_0) = \gamma(0) = x_0$.

Portanto,

$$f(x_0) = f(\gamma(0)) = f(\varphi(0,0,x_0))$$

$$= \frac{\partial}{\partial t}(\varphi(t,0,x_0))\Big|_{t=0}$$

$$\therefore \; f(x) = \frac{\partial}{\partial t}\varphi_t(x)\Big|_{t=0}, \; \forall x \in \Omega.$$



Ficou faltando:

↪ na 1.14, provar que $\bar{d} < +\infty$ e que $\bar{d}(x_n, x) \to 0$.

↳ fica faltando o exercício 1.16.