Universidade Federal Fluminense

Unique ergodicity of the horocycle flow via hyperbolic dynamics

Odylo Abdalla Costa

Niterói

Março 2022

Unique ergodicity of the horocycle flow via hyperbolic dynamics

Odylo Abdalla Costa

Dissertação submetida ao Programa de Pós-Graduação em Matemática da Universidade Federal Fluminense como requisito parcial para a obtenção do grau de Mestre em Matemática.

Orientadores:
Prof. Bruno Santiago (UFF) e Prof. Sébastien Alvarez (Udelar)

Niterói

Março 2022

Ficha catalográfica automática - SDC/BIME
Gerada com informações fornecidas pelo autor

C837u Costa, Odylo Abdalla
Unique ergodicity of the horocycle flow via hyperbolic dynamics / Odylo Abdalla Costa ; Bruno Rodrigues Santiago, orientador ; Sébastien Alvarez, coorientador. Niterói, 2022.
98 f. : il.

Dissertação (mestrado)-Universidade Federal Fluminense, Niterói, 2022.

DOI: http://dx.doi.org/10.22409/PPGMAT.2022.m.12126876713

1. Sistemas Dinâmicos. 2. Teoria Ergódica. 3. Fluxos Geodésicos. 4. Fluxos Anosov. 5. Produção intelectual. I. Santiago, Bruno Rodrigues, orientador. II. Alvarez, Sébastien, coorientador. III. Universidade Federal Fluminense. Instituto de Matemática e Estatística. IV. Título.

CDD -

Bibliotecário responsável: Debora do Nascimento - CRB7/6368

Dissertação de Mestrado da Universidade Federal Fluminense

por

Odylo Abdalla Costa

apresentada ao Programa de Pós-Graduação em Matemática como requesito parcial para a obtenção do grau de

Mestre em Matemática

Título da tese:

Unique ergodicity of the horocycle flow via hyperbolic dynamics

---

Defendida publicamente em 17 de março de 2022.

Diante da banca examinadora composta por:

Bruno Santiago	UFF	Orientador
Sébastien Alvarez	Udelar	Coorientador
Katrin Gelfert	UFRJ	Examinador
Alejandro Kocsard	UFF	Examinador
Rafael Potrie	Udelar	Examinador

DECLARAÇÃO DE CIÊNCIA E CONCORDÂNCIA DO(A) ORIENTADOR(A)

Autor(a) da Dissertação: Odylo Abdalla Costa
Data da defesa: 17/03/2022
Orientadores: Bruno Santiago e Sébastien Alvarez

Para os devidos fins, declaro **estar ciente** do conteúdo desta **versão corrigida** elaborada em atenção às sugestões dos membros da banca examinadora na sessão de defesa do trabalho, manifestando-me **favoravelmente** ao seu encaminhamento e publicação no **Repositório Institucional da UFF**.

Niterói, 17/03/2022.

Bruno Santiago

Nome do orientador(a)

AGRADECIMENTOS

Esta dissertação é a conclusão de um trabalho que começou no fim da minha graduação, ainda em 2019, quando o professor Bruno Santiago me chamou para estudar e aprender um pouco de Teoria da Medida. Foram três anos importantíssimos na minha vida e eu aproveito esse momento para agradecer algumas pessoas que foram fundamentais neles.

Começo agradecendo ao próprio Bruno, a quem devo muito. Suas paciência e dedicação infinitas e seu entusiasmo único são os principais responsáveis pelo aluno que fui durante o mestrado e que sou hoje. Devo a ele também a decisão de permanecer na UFF para o mestrado e o contato e a orientação do professor Sébastien Alvarez.

Ao Sébastien, estendo os agradecimentos feitos ao Bruno, mas aproveito para agradecer pelo carinho e elegância na hora de conversar sobre Matemática: sempre com argumentos cuidadosamente elaborados e (mais uma vez) de uma elegância ímpar.

Ainda agradecendo à professores, agradeço a professores que ministraram cursos que, além de fundamentais para que eu pudesse escrever essa dissertação, foram marcantes pela excelência: aos professores Nils Martin Andersson, Alejandro Kocsard, Andrés Koropecki, Isabel Lugão Rios, Rafael Potrie, Hernán Maycol Falla Luza e Rafael Ruggiero, além dos próprios Bruno e Sébastien, meu muito obrigado.

Gostaria de fazer um agradecimento em separado ao professor Daniele Sepe. Não só pela disponibilidade de conversar comigo e me fornecer referências, ouvir minhas ideias e explicar argumentos, em certo sentido básicos, que eu desconhecia, devo agradecer a ele pela leitura extremamente cuidadosa dos argumentos sobre a interpretação Hamiltoniana dos fluxos geodésicos.

Com respeito à parte administrativa do meu mestrado, agradeço aos também professores Max Souza e Javier Ribón Herguedas por seu auxílio enquanto coordenadores da PGMAT-UFF. Faço um agradecimento especial à Tayene Sena e à Jacqueline Garcez, secretárias do PGMAT. Sem elas, tenho certeza, todos os trâmites referentes ao mestrado beirariam o impossível. Muito obrigado pela disponibilidade infinita.

Agradeço aos meus colegas da UFF, em particular aos amigos Matheus Manso Del Valle, Danilo Machado Tereza, João Rodriguez Marcondes, Luis Gustavo Vazquez e Luana Aquino por sua parceria durante as disciplinas. Também tenho que agradecer aos meus amigos da PUC-RJ: Camila Crispim, Marcelo Durães e Aline de Melo, por me explicar e me ouvir falar de Matemática, em especial em Sistemas Dinâmicos e Teoria Ergódica. Nossas conversas durante o último ano foram fundamentais para o que escrevo aqui.

A pandemia tornou os dois últimos anos difíceis para todos nós. No meu caso em particular, não teria sido possível passar por um mestrado dentro dela sem a companhia, presença e suporte de minha família: Christine, João, Lucy e Maria. Obrigado por estarem sempre ao meu lado e por serem um exemplo que busco seguir.

Preciso agradecer em separado à Caroll. Sua presença, companhia, paciência e amor diários foram combustíveis para os bons e maus momentos. Os últimos anos com ela foram de um colorido inigualável e, se um décimo dessa alegria se refletir no texto que segue, terá melhorado ele para além do que eu mesmo poderia ter feito.

Por fim, agradeço à CAPES e à FAPERJ. À CAPES pelo apoio via uma bolsa de mestrado, vigente de 01/12/2019 a 31/05/2021 (processo de número: 88887.469299/2019-00) e à FAPERJ pelo apoio via uma bolsa Mestrado Nota 10, vigente desde o final da vigência da bolsa da CAPES (processo de número 200.474/2021 (261807)). Sem o fomento das duas agências não teria conseguido realizar o Mestrado. Meu muito obrigado.

O presente trabalho foi realizado com apoio da Coordenação de Aperfeiçoamento de Pessoal de Nível Superior - Brasil (CAPES) - Código de Financiamento 001.

RESUMO

A unicidade ergódica do fluxo horocíclico no fibrado tangente unitário de uma superfície compacta com curvatura negativa é conhecida desde Furstenberg [Fur73]. Esta dissertação de mestrado, sob a orientação dos professores Bruno Santiago (UFF) e Sébastien Alvarez (Udelar), apresenta uma prova alternativa desse resultado, que combina um teorema devido a Coudène [Cou09] sobre teoria ergódica em espaços métricos, e um resultado em dinâmica hiperbólica, devido a Plante [Pla72], que caracteriza quando a folheação instável forte de um fluxo Anosov transitivo é minimal.

Palavras-chave: Sistemas Dinâmicos; Teoria Ergódica; Fluxos Geodésicos; Fluxos Anosov; Fluxo Horocíclico.

ABSTRACT

The unique ergodicity of the horocycle flow in the unit tangent bundle of a compact surface with constant negative curvature is known since Furstenberg [Fur73]. This master's thesis, under the supervision of professors Bruno Santiago (UFF) and Sébastien Alvarez (Udelar), presents an alternative proof of this result, which combines a theorem due to Coudène [Cou09] on ergodic theory on metric spaces and a result from hyperbolic dynamics, due to Plante [Pla72], that characterizes when the strong unstable foliation of a transitive Anosov flow is minimal.

Keywords: Dynamical Systems; Ergodic Theory; Geodesic Flows; Anosov Flows; Horocycle Flows.

List of Figures

Contents

CHAPTER 1

Introduction

Dynamical Systems seek the understanding of the structure of the orbits of systems as it evolves with time. A very classical area of research within Dynamical System focus on the study of systems where such evolution in time occur over a continuous parameter. The idea of such a system is formalized in the concept of a ***flow***. Precisely, a flow is a map $\varphi\colon \mathbb{R} \times \mathcal{M} \to \mathcal{M}$ such that $\varphi(0,x) = x$ for all $x \in \mathcal{M}$ and such that $\varphi(s, \varphi(t,x)) = \varphi(s+t, x)$ for all $x \in \mathcal{M}$ and $s, t \in \mathbb{R}$. Given a flow φ on $\mathcal{M}$, one can wonder what it does for each point $x \in \mathcal{M}$ by studying its **orbit**:

$$\mathcal{O}_\varphi(x) \coloneqq \{\varphi(t,x) \in \mathcal{M} \mid t \in \mathbb{R}\}.$$

Unfortunately, the problem of studying orbits of flows is too abstract to ask for precise answers. Hence, it is useful to require more hypotheses to the problem, for example on the set $\mathcal{M}$ or on the flow φ. One such possible fashion is to demand geometric properties on $\mathcal{M}$ and that is what we do most of the times here.

Given a closed and connected Riemannian manifold $(M, \langle \cdot, \cdot \rangle)$ and a point $x \in M$, there exists a flow defined on the unit tangent bundle T^1M of M, that comes from its geometry, called the **geodesic flow**. This flow consists in, initiating at the point x, to travel trough the geodesic that leaves x with a prescribed direction $v \in T_xM$. With those restrictions, i.e., $\mathcal{M} = T^1M$ being the unit tangent bundle of a closed and connected Riemannian manifold and g the geodesic flow, what can we say about the orbit $\mathcal{O}_g(x,v)$ of a point $(x,v) \in T^1M$?

As we will see in Chapter 3, if M has constant negative seccional curvature, the geodesic flow has very remarkable characteristic: hyperbolicity. In particular, the tangent bundle TT^1M admits an invariant splitting

$$TT^1M = E^s \oplus \langle X \rangle \oplus E^u,$$

where X is the vector field tangent to the flow and the subbundle E^s contracts vectors exponentially fast and the subbundle E^u expands vectors also exponentially fast.

Those spaces are integrable, in the sense that for each $p = (x,v) \in T^1M$ there are submanifolds $W^s(p)$ and $W^u(p)$ on T^1M extremely related with the dynamics presented by the geodesic flow, such that $T_x(W^s(p)) = E^u(p)$ and $T_x(W^u(p)) = E^u(p)$, for each $p \in T^1M$. Whenever M is a surface (of genus at least 2 for it to have a metric of negative curvature), the subbundles E^s and E^u have dimension 1, and also the submanifolds W^s and W^u.

Since those submanifolds are 1-dimensional we can parametrize them by what is called the horocycle flow: $h_s\colon T^1M \to T^1M$. What is the dynamics of this flow? Does it interact with the geodesic flow on

T^1M? In fact, in the setting of constant negative curvature, one has:

$$g_t \circ h_s = h_{se^{-t}} \circ g_t, \tag{1.1}$$

for every $t, s \in \mathbb{R}$.

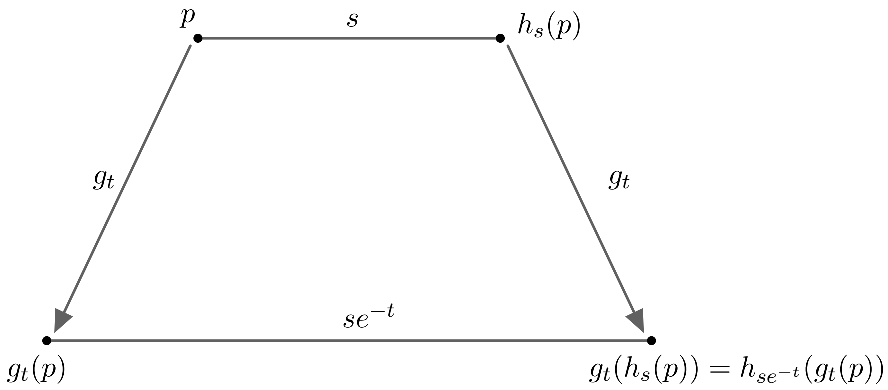

Figure 1.1: Relation between h_s and g_t for $t < 0$.

In particular, equation 1.1 states that g_t moves points in the orbit of h_s by a exponential rate in time. Precisely, if we fix $p \in T^1M$, $s \in \mathbb{R}$ and $t < 0$, the geodesic flow g_t send a segment of orbit h_s of size s in time to a segment of size se^{-t}.

The fact that the horocycle flow parametrizes the stable/unstable manifold of the geodesic flow and from this very particular equation between them suggests that it may be interesting to study the dynamics of h_s. In general, this flow may not have the hyperbolic properties that one may be used to. For example, horocycle flows on compact negatively curved surfaces are examples what are called *parabolic flows*[1]. Therefore, one may turn its attention to the statistical properties of it. The main result of this sort presented here is that the horocycle flow on a compact surface with constant negative curvature is uniquely ergodic.

The text is structured as follows: in Chapter 2 we give a brief introduction on topics in Dynamical Systems and in Ergodic Theory that will be needed throughout the text. In Chapter 3 we take a quick glance at the background needed on both Anosov and geodesic flows. In particular, we present the concept of Anosov flow, and exhibit two examples of it: the suspension flow of an Anosov diffeomorphism and the geodesic flow on a compact negatively curved manifold. Moreover, using tools from symplectic geometry, we prove that these two examples are distinct, meaning that a geodesic flow can never be the suspension of a diffeomorphism on a compact manifold.

Next, in Chapter 4, we establish a topological property of horocycles flows on compact negatively curved surfaces: in this setting, the horocycle flow is a minimal flow, meaning that all its orbits are dense. Instead of giving the classical proof by Hedlund [Hed36], we make a broad walk and present a proof of a more general result by Plante [Pla72], and that uses the hyperbolicity of the geodesic flow:

Theorem (Plante)**.** *The weak and strong stable (unstable) foliations of a transitive Anosov flow on a closed and connected Riemannian manifold are always minimal, unless the flow is the suspension of an Anosov diffeomorphism.*

[1]For a brief description of parabolic flows, see [Ulc21].

Since we will have proved in Chapter 3 that geodesic flows can never be suspensions, we obtain the minimality of the horocycle flow.

Finally, in Chapter 5 we prove the unique ergodicity of the horocycle flow. This result was first obtained by Furstenberg [Fur73]. However, we present here a proof given by Yves Coudène in [Cou09].

Coudène's theorem deals with measures that decomposes on special way: we say a measure μ has a **local product structure** if, in a small enough foliated chart $V = L \times I \subseteq M$, with $L \in \mathcal{F}$, the measure μ can be disintegrated, up to renormalization of the measures, as

$$\int_V f \, d\mu = \int_I \int_L f(x, s) \, d\nu_s \, ds,$$

for all $f \in C^0(M)$. Here, ds is the Lebesgue measure on I and the ν_s are probability measures on L that vary measurably on s.

With this definition we can state Coudène's theorem:

Theorem (Coudène)**.** *Let $g_t \colon M \to M$ be an Anosov flow on a compact Riemannian manifold such that the stable foliation $W^s(x)$ has constant dimension equal 1. Suppose, moreover, that the stable foliation W^s is parametrized by a continuous flow $h_s \colon M \to M$, that the volume measure μ on M is invariant under both flows g_t and h_s, that μ has local product structure and that*

$$g_t \circ h_s = h_{se^{-t}} \circ g_t,$$

for every $t, s \in \mathbb{R}$.

Then, if h_s has a dense orbit, it is uniquely ergodic.

Besides being a short and beautiful proof of a classical result, this theorem has a very notable feature: the way it is stated, the proof still works in general settings, such as the partially hyperbolic one.

CHAPTER 2

Some preliminaries in Dynamics and in Ergodic Theory

This master thesis is about a special class of flows. In this chapter we give a brief introduction on the Dynamics and the Ergodic Theory of flows with some regularity.

Throughout the text, (M, g) will be closed (compact without boundary) and connected Riemannian manifold. As presented at the Introduction, a flow $\varphi\colon \mathbb{R} \times M \to M$ is a map satisfying:

- $\varphi(0, x) = x$ for all $x \in M$;
- $\varphi(s, \varphi(t, x)) = \varphi(s + t, x)$ for all $x \in M$ and $s, t \in \mathbb{R}$.

Unless we explicitly say otherwise, we will always assume the flow φ to be of class C^r, with $r \geq 1$. In particular, for each $t \in \mathbb{R}$, the map $\varphi_t\colon M \to M$ defined by $\varphi_t(x) = \varphi(t, x)$, always is of class C^r.

The existence of flows on manifolds are intimately related to the existence of vector fields, as next example tell us.

Example 1 (Flows and vector fields)**.** *Consider a vector field $X \in \mathfrak{X}^r(M)$. Then, the Fundamental Theorem of ODE's guarantees that, through each point $p \in M$, the initial value problem*

$$\begin{cases} x'(t) = X(x(t)) \\ x(0) = p \end{cases}$$

admits a unique solution $\gamma_p\colon \mathbb{R} \to M$. Moreover, the map $\varphi\colon \mathbb{R} \times M \to M$ defined by $(t, p) \mapsto \gamma_p(t)$ is a flow of class C^r such that

$$\frac{\partial \varphi(t, p)}{\partial t} = X(\varphi(t, p)).$$

Reciprocally, to each flow φ_t on a manifold M, there is a vector field X that it integrates: one just has to define $X(p) = X(\varphi_0(p)) = \frac{\partial \varphi(0,p)}{\partial t}$, for each $p \in M$.

A general statement, as well as a proof, of the Fundamental Theorem of ODE's can be found in [Lee13], as Theorem 9.12, p. 212. Also, the fact that flows on compact manifolds are complete, i.e., are well-defined over $\mathbb{R}$, is proved in the same text: see Corollary 9.17, p. 216.

We now give a concrete example of a flow on a compact manifold. To do so, we explicit the construction of the quotient $\mathbb{T}^n = \mathbb{R}^n / \mathbb{Z}^n$. Define, on $\mathbb{R}^n$, the following equivalence relation: we say two points $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$ in $\mathbb{R}^n$ are equivalent if and only if their difference is an integer vector. More explicitly,

$$(x_1, \dots, x_n) \sim (y_1, \dots, y_n) \iff (x_1 - y_1, \dots, x_n - y_n) \in \mathbb{Z}^n.$$

We denote by $[x]$ or $[(x_1, \dots, x_n)]$ the equivalence class of the point $x = (x_1, \dots, x_n)$. Finally, we define $\mathbb{T}^d$ to be the quotient of $\mathbb{R}^n$ by this equivalence relation: $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$.

Note that $\mathbb{T}^n$ is an abelian group with the operation:

$$[(x_1, \dots, x_n)] + [(y_1, \dots, y_n)] = [(x_1 + y_1, \dots, x_n + y_n)].$$

Example 2 (Linear flow on $\mathbb{T}^n$)**.** *Let $\theta = (\theta_1, \dots, \theta_n) \in \mathbb{R}^n$ be a fixed vector and let $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ be the $n-$dimensional torus endowed with the volume measure μ. Define the* ***linear flow*** *φ_t* ***on*** *$\mathbb{T}^n$* ***in the direction of*** *θ as the map $\varphi_t \colon \mathbb{T}^n \to \mathbb{T}^n$ such that to each $[x] = [(x_1, \dots, x_n)] \in \mathbb{T}^n$ associates*

$$\varphi_t(x) = [x + t\theta].$$

In this example, the linear flow φ_t is the solution of the following ODE on $\mathbb{T}^n$:

$$\frac{dx}{dt} = \theta.$$

As presented in the Introduction, given a flow φ on M, we want to know what happens to its orbits:

$$\mathcal{O}_\varphi(x) := \{\varphi(t, x) \in M \mid t \in \mathbb{R}\},$$

for each $x \in M$.

Since we are mainly dealing with invertible systems, it makes sense to brake the orbit of each point $x \in M$ into two subsets: the ***positive semi-orbit*** and the ***negative semi-orbit*** by the flow φ. Respectively, they are defined as follows:

- $\mathcal{O}^+_\varphi(x) := \{\varphi(t, x) \in M \mid t \geq 0\}$;
- $\mathcal{O}^-_\varphi(x) := \{\varphi(t, x) \in M \mid t \leq 0\}$.

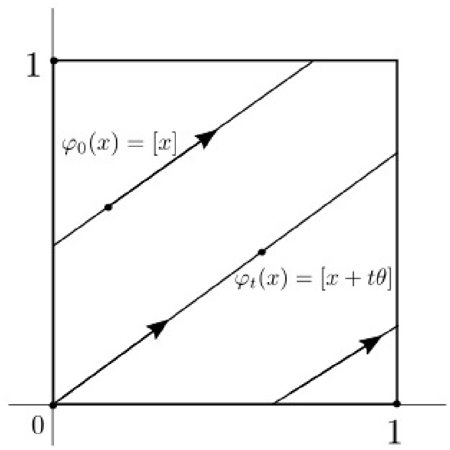

Figure 2.1: The flow $\varphi_t(x) = [x + t\theta]$ on $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$.

With these definitions, $\mathcal{O}_\varphi(x) = \mathcal{O}_\varphi^-(x) \cup \mathcal{O}_\varphi^+(x)$.

If, for a point $p \in M$ there exists a time $T \in \mathbb{R}$ such that $\varphi_T(p) = p$, we call the point p a ***periodic point*** for φ. Also in this setting, and we say the orbit of p is ***closed*** and if $\tau \in \mathbb{R}$ is such that $\varphi_\tau(p) = p$ and for all $0 < t < \tau$, we have $\varphi_t(p) \neq p$, then we say the orbit of p closed of period τ. The set of all periodic points p for φ is denoted by $\mathrm{Per}\,(\varphi)$.

In general, flows can have plenty, few, or even none periodic orbits. Even in the simple setting of Example 2, the orbits of point through the flow behave very differently depending on the direction vector θ:

Proposition 1. *Consider $\theta = (\theta_1, \dots, \theta_n) \in \mathbb{R}^n$. If each θ_i is rational, say $\theta_i = \frac{p_i}{q_i}$ with $p_i, q_i \in \mathbb{Z}$, $q_i \neq 0$ and $\gcd(p_i, q_i) = 1$ for each $i = 1, \dots, n$, then each point $x \in \mathbb{T}^n$ is periodic.*

Proof. Indeed, consider $T = \mathrm{lcm}\,(q_1, \dots, q_n)$. Then,

$$\varphi_T(x) = [x + T\theta] = [x],$$

for all $x \in \mathbb{T}^n$. □

On the opposite direction of the above proposition, if α is a irrational number, then the linear flow in the direction of $\theta = (\alpha, 0, \dots, 0)$ has no periodic points: for each $x \in \mathbb{T}^n$, the orbit $\varphi_t(x)$ remains in a vertical circle, on which the dynamics is an irrational rotation by α. Hence, this linear flow has no periodic orbit.

During the text we will focus on two different ways of studying the orbit of point x by a flow φ: a topological and a measure theoretic one. At this point, however, we make a brief detour in order to look at the orbit of regular flows via foliation theory.

2.1 Dynamics and foliations

This section has as its main objective to define foliations and to present their relation to flows with certain regularity. In particular, we want to stress out the fact that the orbits of the linear flow on $\mathbb{T}^n$ produce a foliation with several dynamical properties of great interest. Through this Section we follow mainly [CN13].

Definition 1 (Foliation)**.** *Let M be a smooth manifold of dimension m. A C^r foliation of dimension n in M is a C^r atlas $\mathcal{F}$ on M which is maximal with the following properties:*

(a) *If (U, φ) is a chart in $\mathcal{F}$, then $\varphi(U) = U_1 \times U_2 \subset \mathbb{R}^n \times \mathbb{R}^{m-n}$ where U_1 and U_2 are open discs in $\mathbb{R}^n$ and $\mathbb{R}^{m-n}$, respectively;*

(b) *If (U, φ) and (V, ψ) are charts in $\mathcal{F}$ such that $U \cap V \neq \emptyset$ then the change of coordinates map $\psi \circ \varphi^{-1} \colon \varphi(U \cap V) \to \psi(U \cap V)$ is of the form*

$$\psi \circ \varphi^{-1}(x, y) = (h_1(x, y), h_2(y)),$$

where h_1 and h_2 are C^r diffeomorphisms with $(x, y) \in (U_1 \cap V_1) \times (U_2 \cap V_2)$.

Whenever M admits such an atlas $\mathcal{F}$, we say that M is foliated by $\mathcal{F}$, or that $\mathcal{F}$ is a foliated structure of dimension n and class C^r on M, and call the charts $(U, \varphi) \in \mathcal{F}$ foliation charts.

Example 3. *Our first example of foliation is the example o a foliation defined by a submersion.*

Let $f\colon M \to N$, a C^r submersion between manifolds M and N of dimensions m and n, respectively. Given a point $p \in M$ we can use the local form of the submersions to obtain local charts (U, φ) on M and (V, ψ) on N, such that $p \in U$, $f(p) \in V$, $\varphi(U) = U_1 \times U_2 \subseteq \mathbb{R}^{m-n} \times \mathbb{R}^n$, and $\psi(V) = V_2 \supset U_2$ and the composition $\psi \circ f \circ \varphi^{-1}\colon U_1 \times U_2 \to U_2$ has the form of a projection π_2 to second coordinate in $\mathbb{R}^m = \mathbb{R}^{m-n} \times \mathbb{R}^n$: $\psi \circ f \circ \varphi^{-1}(x, y) = y$, as shown in Figure 2.2 below.

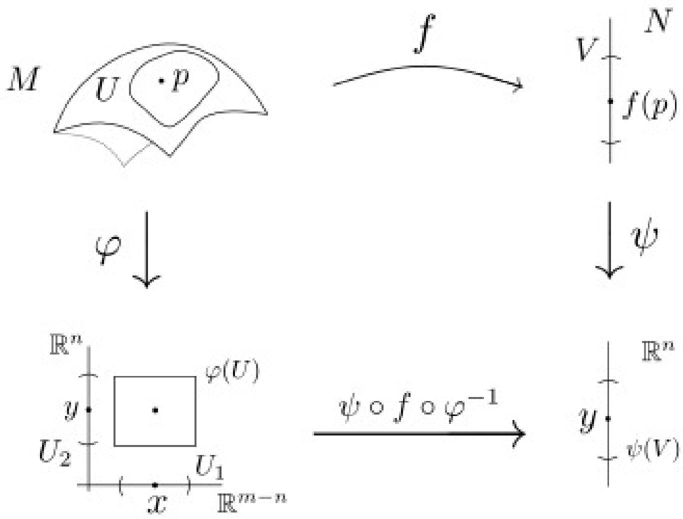

Figure 2.2: Local form of the submersions.

From that we obtain a C^r-foliation $\mathcal{F}$ of dimension n on M: for the foliated charts we choose, for each point $p \in M$, the chart (U, φ) which satisfies the local form of the submersions with some local chart (V, ψ) over $f(p)$.

To check that $\mathcal{F}$ is indeed a foliation we only need to see the condition of compatibility of the charts: let (U, φ) and $(\widetilde{U}, \widetilde{\varphi})$ be charts in $\mathcal{F}$ such that $U \cap V \neq \emptyset$. So we must prove that, on $\varphi(U \cap \widetilde{U})$, one can write:

$$\widetilde{\varphi} \circ \varphi^{-1}(x, y) = (h_1(x, y), h_2(y)).$$

In order to do that, pick $p \in U \cap \widetilde{U}$ and let (V, ψ) and $(\widetilde{V}, \widetilde{\psi})$ be charts on N over $f(p)$ such that, on $\varphi(U \cap \widetilde{U})$ and on $\widetilde{\varphi}(U \cap \widetilde{U})$ we have:

$$\psi \circ f \circ \varphi^{-1}\big|_{\varphi(U)} = \pi_2\big|_{\varphi(U)} \tag{2.1}$$

and

$$\widetilde{\psi} \circ f \circ (\widetilde{\varphi})^{-1}\Big|_{\widetilde{\varphi}(\widetilde{U})} = \pi_2\big|_{\widetilde{\varphi}(\widetilde{U})}. \tag{2.2}$$

Therefore, if we write $\widetilde{\varphi} \circ \varphi^{-1}\colon \varphi\left(U \cap \widetilde{U}\right) \to \widetilde{\varphi}\left(U \cap \widetilde{U}\right)$ as

$$\widetilde{\varphi} \circ \varphi^{-1}(x, y) = (h_1(x, y), h_2(x, y)),$$

we have:

$$
\begin{aligned}
h_2(x,y) &= \pi_2 \circ \widetilde{\varphi} \circ \varphi^{-1}(x,y) \\
&= \widetilde{\psi} \circ f \circ \widetilde{\varphi}^{-1} \circ \widetilde{\varphi} \circ \varphi^{-1}(x,y) && \text{(by 2.2)} \\
&= \widetilde{\psi} \circ f \circ \varphi^{-1}(x,y) \\
&= \widetilde{\psi} \circ \psi^{-1} \circ \psi \circ f \circ \varphi^{-1}(x,y) \\
&= \widetilde{\psi} \circ \psi^{-1} \circ \pi_2(x,y) && \text{(by 2.1)} \\
&= \widetilde{\psi} \circ \psi^{-1}(y),
\end{aligned}
$$

meaning that we can write $h_2(x,y)$ simply as $h_2(y)$, as we wished. Hence, (U,φ) is a foliated chart of the foliated structure $\mathcal{F}$ of dimension n and class C^r on M.

Definition 2. *Given a C^r foliation $\mathcal{F}$ of dimension n on a $m-$dimensional smooth manifold M (where $0 < n < m$). Consider a local chart $(U,\varphi) \in \mathcal{F}$ such that $\varphi(U) = U_1 \times U_2 \subseteq \mathbb{R}^n \times \mathbb{R}^{m-n}$. We call the sets of the form $\varphi^{-1}(U_1 \times \{c\})$, with $c \in U_2$, the* ***plaques of*** *U (or of $\mathcal{F}$).*

A ***path of plaques of*** *$\mathcal{F}$ is a sequence $\alpha_1, \ldots, \alpha_k$ of plaques of $\mathcal{F}$ such that $\alpha_j \cap \alpha_{j+1} \neq \emptyset$ for all $j \in \{1, \ldots, k-1\}$. Moreover, since we can cover M by plaques of $\mathcal{F}$, we can define the following equivalence relation on M:*

$$p \sim q \iff \text{there exists a path of plaques } \alpha_1, \ldots, \alpha_k \text{ with } p \in \alpha_1 \text{ and } q \in \alpha_k.$$

The equivalence classes of the relation $\sim$ on M are called ***leaves*** *of the foliation $\mathcal{F}$.*

Notice that, given a local chart $(U,\varphi) \in \mathcal{F}$ such that $\varphi(U) = U_1 \times U_2 \subseteq \mathbb{R}^n \times \mathbb{R}^{m-n}$ as above, if we fix a point $c \in U_2$, the map $\varphi^{-1}\big|_{U_1 \times \{c\}} \colon U_1 \times \{c\} \to U$ is a C^r embedding. Remembering that U_1 is a open disc, the plaques are path-connected $n-$dimensional C^r submanifolds of M.

Therefore, if p and q in M are in the same leaf of $\mathcal{F}$, there is a path of plaques connecting the two and, moreover, there is a continuous path connecting them because $\alpha_j \cap \alpha_{j+1} \neq \emptyset$ for all $j \in \{1, \ldots, k-1\}$ and the plaques are path-connected.

Example 4. *In Example 3 the leaves are the connected components of the level sets $f^{-1}(c)$, where $c \in N$.*

Example 5. *Let $f \colon \mathbb{R}^3 \to \mathbb{R}$ be a submersion defined by*

$$f(x,y,z) = \alpha(x^2 + y^2) \cdot e^z,$$

where $\alpha \colon \mathbb{R} \to \mathbb{R}$ is a C^∞ function such that $\alpha(1) = 0$, $\alpha(0) = 1$ and if $t > 0$ then $\alpha' < 0$.

Using the construction of the Example 3, let $\mathcal{F}$ be the foliation of $\mathbb{R}^3$ whose leaves are the connected components of the submanifolds $f^{-1}(c)$, for $c \in \mathbb{R}$.

The leaves of $\mathcal{F}$ are of three types, all ruled by the relation with the solid cylinder

$$C = \{(x,y,z) \in \mathbb{R}^3 \mid x^2 + y^2 \leq 1\},$$

in the following way:

(i) *the boundary of C, i.e., $\partial C = \{(x,y,z) \in \mathbb{R}^3 \mid x^2 + y^2 = 1\}$ is a leaf of $\mathcal{F}$;*

(ii) *outside C, i.e., on the set of points $(x, y, z) \in \mathbb{R}^3$ such that $x^2 + y^2 > 1$, the leafs of $\mathcal{F}$ are all homeomorphic to cylinders;*

(iii) *finally, in the interior of C, i.e., on the set of points $(x, y, z) \in \mathbb{R}^3$ such that $x^2 + y^2 < 1$, the leafs of $\mathcal{F}$ are all homeomorphic to $\mathbb{R}^2$ by a parametrization $\sigma\colon \mathbb{D}^2 \to \mathbb{R}^3$ from the disk $\mathbb{D}^2 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1\}$ to $\mathbb{R}^3$, defined by:*

$$\sigma(x, y) = \left(x, y, \log\left(\frac{c}{\alpha(x^2 + y^2)}\right)\right).$$

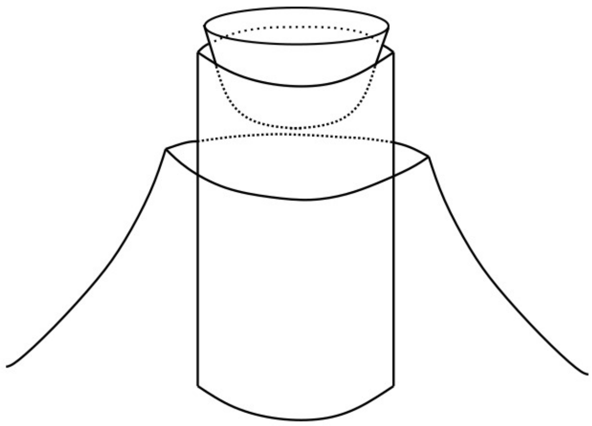

Figure 2.3: Example of foliation coming from a submersion.

The next example is the main example of this Section and has a very dynamical nature.

Example 6. *Foliations arising from vector fields without singularities.*

Let X be a C^r $(r \geq 1)$ vector field without singularities on a compact manifold M (with $\dim M = m$). As we have seen in Example 1, associated to X we have a flow $\varphi(t, x)$ such that

$$X(\varphi(t, x)) = \frac{\partial \varphi(t, x)}{\partial t}$$

for every $(t, x) \in \mathbb{R} \times M$.

Let $i\colon \mathbb{B}^{m-1}(0) \to M$ be an embedding of a small $m - 1$ disk around $0 \in \mathbb{R}^m$, such that $i(0) = p$, that is transverse to X everywhere. Since $X(p) \neq 0$, for $\varepsilon > 0$, the map

$$\Phi\colon \mathbb{B}^{m-1}(0) \times (-\varepsilon, \varepsilon) \to M$$

defined by

$$\Phi(x, t) = \varphi(t, i(x))$$

has maximal rank at $(0, 0) \in \mathbb{B}^{m-1}(0) \times (-\varepsilon, \varepsilon)$.

By the Inverse Mapping Theorem, there is a neighborhood $V \subset M$ around p such that $\Phi^{-1}\big|_V$ is a diffeomorphism between V and a product neighborhood $\tilde{\mathbb{B}}^{m-1}(0) \times (-\varepsilon', \varepsilon') \subseteq \mathbb{B}^{m-1}(0) \times (-\varepsilon, \varepsilon)$ of $(0, 0)$. This is a local chart for the one-dimensional foliation on M defined by the curves $t \mapsto \varphi_t(x)$, the integral curves of X.

Therefore, from a regular vector X on M, we obtain a one-dimensional foliation $\mathcal{F}$ whose leafs are the integral curves of X.

Example 7 (Linear flow on $\mathbb{T}^n$)**.** *A particular case of the previous example, is the case where $M = \mathbb{T}^n$ and $X(x) = \theta$ for all $x \in \mathbb{T}^n$, where $\theta \in \mathbb{R}^n$.*

From Example 2, we know that the solutions of the ODE $\frac{dx}{dt} = X(x)$ is $\varphi_t(x) = [x + t\theta]$. Hence, in the particular case of the foliation obtained from X, the leafs are

$$L_x = \{\varphi_t(x) \mid t \in \mathbb{R}\}.$$

In the next section, Section 2.2 we will study the topology of this foliations that arises from a dynamical systems X as a subset of $\mathbb{T}^n$.

For now, we comment on how we could extend the relation between foliations on manifolds M and higher dimensional analogues of vector fields.

Definition 3. *A field of $k-$planes on a manifold M is a map $P\colon M \to G_k(TM)$[1] which associates each point $x \in M$ a $k-$dimensional vector subspace $P(x) \subset T_xM$. In the particular case of $k = 1$, we call the map P a **line field**.*

We say a $k-$plane field P on M is of class C^r if, for every $q \in M$, there exist k vector fields $X_1, \ldots, X_k$ defined in a neighborhood V of q and of class C^r, and such that $\{X_1(x), \ldots, X_k(x)\}$ is a basis for $P(x)$ for every $x \in V$.

Definition 4. *Given a $k-$plane field P on M, we call a submanifold $N \subset M$ an **integral manifold of** P if $T_xN = P(x)$ for every $x \in N$. We say P is **integrable** if there exists a foliation $\mathcal{F}$ such that, for every point $x \in M$ there exists a leaf $\mathcal{F}(x)$ of $\mathcal{F}$ such that $T_x(\mathcal{F}(x)) = P(x)$. Moreover, we say that P is **uniquely integrable** if the foliation above is unique.*

Definition 5. *We say a plane field P is **completely integrable** if, given two vector fields X and Y such that, for each $q \in M$, if $X(q)$ and $Y(q)$ are in $P(q)$, then $[X, Y](q) \in P(q)$, where $[\cdot, \cdot]$ is the Lie bracket on M.*

Finally, we present a theorem of Frobenius that generalizes to plane fields the existence of tangent foliations:

Theorem. *Let P be a C^r $k-$plane field (for $k \geq 1$) on M. If P is completely integrable, then there exists a C^r foliation $\mathcal{F}$ of dimension k on M such that $T_q(\mathcal{F}) = P(q)$ for all $q \in M$. Conversely, if $\mathcal{F}$ is a C^r ($r \geq 2$) foliation and P is the tangent plane field to $\mathcal{F}$, then P is uniquely integrable.*

2.2 Topological Dynamical Systems

In this section we present some facts and definitions, that deal mostly with asymptotic features of orbits for continuous flows.

Definition 6 (Limit set)**.** *The $\omega-$**limit set** of a point $x \in M$ is the closed set of limit points of the positive semi-orbit*

$$\omega(x) = \bigcap_{t \geq 0} \overline{\mathcal{O}^+_\varphi(\varphi_t(x))}.$$

[1] Here $G_k(TM)$ denotes the Grassmannian manifold of $k-$dimensional subspaces of TM.

*Similarly, we define the $\alpha-$**limit set** of a point $x \in M$ as the closed set of limit points of the negative semi-orbit*

$$\alpha(x) = \bigcap_{t \geq 0} \overline{\mathcal{O}_{\varphi}^{-}(\varphi_{-t}(x))}.$$

*Finally, we define the **limit set** of a flow φ as the set:*

$$L(\varphi) = \bigcup_{x \in M} \overline{\omega(x) \cup \alpha(x)}.$$

Definition 7 (Nonwandering set)**.** *A point $x \in M$ is called **nonwandering** for a flow $\varphi_t \colon M \to M$ if for any open set U containing x, and every $T > 0$, there is a $t > T$ such that*

$$\varphi_t(U) \cap U \neq \emptyset.$$

*Naturally, a point is called **wandering** if it is not nonwandering. The set of all nonwandering points for a flow φ_t is called the **nonwandering set** and is denoted by $\Omega(\varphi)$.*

Definition 8 (Transitivity)**.** *We say a continuous flow $\varphi_t \colon M \to M$ is **topologically transitive** (or simply **transitive**) if for every $x \in M$ its positive orbit by the flow is dense on M, i.e., $\overline{\mathcal{O}_{\varphi}^{+}(x)} = M$.*

Proposition 2 (Characterization of transitivity)**.** *Suppose M to be a compact metric space and φ_t a continuous flow on M. Then the following conditions are equivalent:*

(i) *φ_t is transitive (has a dense positive orbit);*

(ii) *φ_t has a dense orbit;*

(iii) *for non-empty open sets $U, V \subset M$ there exists $t \in \mathbb{R}$ such that $\varphi_t(U) \cap V \neq \emptyset$;*

(iv) *for non-empty open sets $U, V \subset M$ there exists $t \geq 0$ such that $\varphi_t(U) \cap V \neq \emptyset$.*

Proof. For a proof see Proposition 1.6.9, p. 80, on [FH19]. □

Next, we follow [FH19] on the review of the example of the linear flow on $\mathbb{T}^n$ to check what are the conditions on θ that makes $\varphi_t(x) = [x + t\theta]$ transitive.

Definition 9. *We say the components of a vector $\theta \in \mathbb{T}^n$ is rationally independent if $k \in \mathbb{Z}^n$ and $\langle k, v \rangle = 0$ then $k = 0$.*

Proposition 3. *A linear flow $\varphi_t(x) = [x + t\theta]$ on $\mathbb{T}^n$ is transitive if, and only if, the components of θ are rationally independent.*

To prove Proposition 3 we need a lemma:

Lemma 1. *Every continuous function invariant by a transitive flow is constant.*

Proof. Suppose $f \colon M \to \mathbb{R}$ is an φ_t-invariant continuous function and let $x \in M$ be such that $\overline{\mathcal{O}_{\varphi}(x)} = M$.

Since f is φ_t-invariant, there is $c \in \mathbb{R}$ such that $f(\mathcal{O}_{\varphi}(x)) = \{c\}$. Since f is continuous, we can write:

$$\{c\} = f(\mathcal{O}_{\varphi}(x)) = f\left(\overline{\mathcal{O}_{\varphi}(x)}\right) = f(M),$$

i.e., f is constant. □

Proof of Proposition 3. First we show that, if θ is not rationally independent, then φ_t is not transitive. The idea here is to construct a non-constant invariant function and then use Lemma 1.

Suppose $\langle k, \theta \rangle = 0$ and that not all k_i's are zero. Next, consider the function $f\colon \mathbb{T}^n \to \mathbb{R}$ defined by

$$f(x) = \sin 2\pi\langle k, x\rangle.$$

The function is well-defined on $\mathbb{T}^n$ by periodicity of $\sin$ and is non-constant since rationally independence implies that not all the k_i's are zero, and then f is not constant. However, it is φ_t-invariant:

$$\begin{aligned} f(\varphi_t(x)) &= \sin 2\pi\langle k, \varphi_t(x)\rangle \\ &= \sin 2\pi\langle k, x + t\theta\rangle \\ &= \sin 2\pi\left(\langle k, x\rangle + t\langle k, \theta\rangle\right) \\ &= \sin 2\pi\langle k, x\rangle \\ &= f(x). \end{aligned}$$

Then, f is a φ_t-invariant non-constant continuous function from $\mathbb{T}^n$ to $\mathbb{R}$ and, hence, φ_t is not transitive. This proves that: if φ_t is transitive, then the θ components of θ are rationally independent.

To prove the reciprocal, will be proven in Section 2.3, after we introduce the notion of invariant measure. □

Next we present another important concept on topological dynamical systems.

Definition 10 (Minimality)**.** *A continuous flow on a compact metric space is called* ***minimal*** *if every orbit is dense.*

As in the case of transitivity, the linear flow provides a variety of examples depending on the choice of θ:

Proposition 4. *A linear flow $\varphi_t(x) = [x + t\theta]$ on $\mathbb{T}^n$ is minimal if, and only if, the components of θ are rationally independent.*

Proof. Of course, if φ_t is minimal, then it is transitive and then the components of θ are rationally independent.

Reciprocally, suppose rationally independence and consequently transitivity of φ_t. Let x be a point on $\mathbb{T}^n$ such that its orbit is dense: $\overline{\mathcal{O}_\varphi(x)} = \mathbb{T}^n$. We claim that, for every other $x' \in \mathbb{T}^n$, we also have $\overline{\mathcal{O}_\varphi(x')} = \mathbb{T}^n$.

Observe that:

$$\begin{aligned} \varphi_t(x') &= x' + t\theta \\ &= (x' - x) + x + t\theta \\ &= (x' - x) + \varphi_t(x), \end{aligned}$$

so that $\mathcal{O}_\varphi(x') = (x' - x) + \mathcal{O}_\varphi(x)$ and then $\overline{\mathcal{O}_\varphi(x')} = (x' - x) + \overline{\mathcal{O}_\varphi(x)}$. Hence, $\overline{\mathcal{O}_\varphi(x')} = \mathbb{T}^n$ if, and only if, $\overline{\mathcal{O}_\varphi(x)} = \mathbb{T}^n$. This proves φ_t is minimal. □

2.3 Ergodic Theory

Definition 11. *Let $(M, \mathcal{B}, \mu)$ be a measure space. We say a flow $\varphi\colon \mathbb{R} \times M \to M$ is* ***measure preserving*** *if for each $t \in \mathbb{R}$:*

- *$\varphi_t\colon M \to M$ is a measurable function;*
- *$\mu(\varphi_t(A)) = \mu(A)$ for every measurable set $A \in \mathcal{B}$.*

Example 8. *Consider, on $\mathbb{T}^n$, the Lebesgue probability measure defined as follows. Let* Leb *be the Lebesgue measure on $\mathbb{R}^n$, $\mathcal{B}$ the Borel $\sigma-$algebra, and $p = \pi\big|_{[0,1]^n}\colon [0,1]^n \to \mathbb{T}^n$ be the restriction to $[0,1]^n$ of the canonical projection of $\mathbb{R}^n$ to $\mathbb{T}^n$, i.e., p is the map defined as $x \mapsto p(x) = [x]$ for every $x \in [0,1]^n$.*

Call a set $B \subseteq \mathbb{T}^n$ measurable on $\mathbb{T}^n$ if $P^{-1}(B)$ is measurable, and define the ***Lebesgue probability measure on*** $\mathbb{T}^n$ *by:*

$$\mu(B) = \mathrm{Leb}\,(p^{-1}(B)),$$

for every measurable set on $\mathbb{T}^n$.

Since the Lebesgue measure Leb *on $\mathbb{R}^n$ is invariant under translation, this measure μ is invariant under the linear flow φ_t on $\mathbb{T}^n$ in the direction of a vector $\theta \in \mathbb{R}^n$.*

As we will see in more generality in Section 3.2, the invariance of the Lebesgue (volume) measure in this context is a consequence of the Liouville theorem (see Theorem 1.3.7, p.21 of [VO16]):

Theorem (Liouville)**.** *Let $\varphi_t\colon M \to M$ be the flow associated to a C^1 vector field X on M. Then, φ_t preserves the volume of M if and only if* $\operatorname{div} X = 0$.

A consequence of the invariance of the Lebesgue measure on $\mathbb{T}^n$ by the linear flow φ_t allow us to follow [FH19] in the proof of the *if* part of Proposition 3, i.e.,

Proposition. *If the components of θ are rationally independent then $\varphi_t(x) = [x + t\theta]$ is transitive.*

Before proving it we need one more lemma:

Lemma 2. *If φ_t is a continuous flow on $\mathbb{T}^n$ and every bounded Lebesgue measurable φ_t-invariant function is constant, then φ_t is transitive.*

Proof. Let O be an open φ_t-invariant set then χ_O its characteristic function is φ_t-invariant. By hypothesis, χ_O will be constant for $\mathrm{Leb}-$almost everywhere. Hence, $\mathrm{Leb}\,(O) = 0$ or $\mathrm{Leb}\,(O) = \mathrm{Leb}\,(\mathbb{T}^n) = 1$, since:

$$\mathrm{Leb}\,(O) = \int_{\mathbb{T}^n} \chi_O \, d\,\mathrm{Leb} = \begin{cases} 0, \text{ if } \chi_O(x) = 0 \text{ for } \mathrm{Leb}-a.e.\ x \in \mathbb{T}^n \\ 1, \text{ if } \chi_O(x) = 1 \text{ for } \mathrm{Leb}-a.e.\ x \in \mathbb{T}^n \end{cases}.$$

In particular, there are no **disjoint** non-empty φ_t-invariant open sets.

Now, let U and V be non-empty open sets on $\mathbb{T}^n$. Hence, the φ_t-invariant open sets:

$$\widetilde{U} = \bigcup_{t \in \mathbb{R}} \varphi_t(U)$$

and

$$\widetilde{V} = \bigcup_{t \in \mathbb{R}} \varphi_t(V)$$

are not disjoint. In particular, there are $t_0, s_0 \in \mathbb{R}$ such that

$$\varphi_{t_0}(U) \cap \varphi_{s_0}(V) \neq \emptyset.$$

Therefore, $\varphi_{t_0 - s_0}(U) \cap V \neq \emptyset$, which proves transitivity of φ_t by Proposition 2. $\square$

*Proof of Proposition 3 - **if** part.* Suppose φ_t is not transitive. Then, by the previous lemma, there exists a bounded Lebesgue φ_t-invariant measurable function $f\colon \mathbb{T}^n \to \mathbb{R}$ that is not constant.

Observe that, being bounded and Lebesgue measurable on $\mathbb{T}^n$, the function $f\colon \mathbb{T}^n \to \mathbb{R}$ admits a Fourier expansion, say:

$$f(x) = \sum_{k \in \mathbb{Z}^n} f_k \cdot e^{2\pi \langle k,x \rangle}.$$

Since f is φ_t-invariant we have:

$$\begin{aligned}
\sum_{k \in \mathbb{Z}^n} f_k \cdot e^{2\pi i \langle k,x \rangle} &= f(x) \\
&= f(\varphi_t(x)) \\
&= f(x + t\theta) \\
&= \sum_{k \in \mathbb{Z}^n} f_k \cdot e^{2\pi i \langle k, x+t\theta \rangle} \\
&= \sum_{k \in \mathbb{Z}^n} f_k \cdot e^{(2\pi i \langle k,x \rangle) + (2\pi i t \langle k,\theta \rangle)} \\
&= \sum_{k \in \mathbb{Z}^n} f_k \cdot e^{2\pi i \langle k,x \rangle} \cdot e^{2\pi i t \langle k,\theta \rangle}.
\end{aligned}$$

Now, since f is not constant, there exists some $k \in \mathbb{Z}^n - \{0\}$ such that $f_k \neq 0$. By uniqueness of the Fourier expansion of f, we then have

$$e^{2\pi \langle k,x \rangle} = e^{2\pi i \langle k,x \rangle} \cdot e^{2\pi i t \langle k,\theta \rangle},$$

so that $e^{2\pi i t \langle k,\theta \rangle} = 1$ for every $t \in \mathbb{R}$. So, we must have $\langle k, \theta \rangle = 0$. This implies that the components of θ are not rationally independent.

So we have shown that if φ_t is not transitive, then the components of θ are not rationally independent. Equivalently, if the components of θ are rationally independent, φ_t is transitive. $\square$

After giving the family of examples of flows that arises from C^1 vector fields X with div $X = 0$, and after studying a little the dynamics of the linear flow on $\mathbb{T}^n$, one could ask a more general question: given a flow on some space, there is some probability measure that is invariant by it. For continuous flows on metrizable compact spaces, the answer is positive.

Theorem 1 (Krylov-Bogoliouboff)**.** *Any continuous flow on a metrizable compact space has an invariant Borel probability measure.*

Proof. See Theorem 3.1.15, p. 161, at [FH19]. $\square$

Poincaré Recurrence Theorem and some of its consequences

The study of systems that preserve measures have several consequences on the theory of Dynamical Systems. One of the most important one due Poincaré:

Theorem 2 (Poincaré Recurrence Theorem)**.** *Let $\varphi\colon \mathbb{R}\times M \to M$ be a measure preserving flow of a probability space $(M, \mathcal{B}, \mu)$. If $E \subseteq M$ is a measurable set with $\mu(E) > 0$ then, for $\mu-$almost every $x \in E$, there a sequence $(t_n)_n$ with $t_n \to +\infty$ such that $\varphi_{t_n}(x) \in E$ for all $n \in \mathbb{N}$.*

Proof. A proof of Poincaré Recurrence Theorem can be found in several books, such as [FH19] (as Theorem 3.2.1 in p. 163). For its discrete version, one can look at [VO16] (Theorem 1.2.1, p. 4). □

As claimed, Poincaré's theorem has a very direct consequence on the dynamics of a flow $\varphi\colon \mathbb{R}\times M \to M$ on a compact metric space M:

Corollary 1 (Poincaré Recurrence Theorem - topological version)**.** *Let M be a compact metric space and $\mathcal{B}$ its Borel $\sigma-$algebra. If $\varphi\colon \mathbb{R}\times M \to M$ is a continuous flow and μ is a $\varphi-$invariant Borel probability measure on M, then*

$$\mu(\overline{\mathrm{Rec}(\varphi)}) = \mu(L(\varphi)) = \mu(\Omega(\varphi)) = 1.$$

In particular, $\mu-$almost every point $x \in M$ is recurrent for φ.

Proof. For a proof see [FH19], Corollary 3.2.2, p. 163. □

Corollary 2 (Birkhoff Recurrence Theorem)**.** *Let M be a compact metric space and $\varphi\colon \mathbb{R}\times M \to M$ a continuous flow on M. Then there exists some point $x \in M$ that is recurrent for φ.*

Proof. Since φ is a continuous flow on a compact metric space, Theorem 1 say that there exists an invariant Borel probability measure. Now, Corollary 1 guarantees that $\overline{\mathrm{Rec}\,(\varphi)} \neq \emptyset$. Hence, $\mathrm{Rec}\,(\varphi) \neq \emptyset$. □

Ergodic Theorems

Poincaré Recurrence Theorem guarantees that, given a set of positive measure, almost every point of the set must return infinitely many times to it. But can we have more information on how it happens? The ergodic theorems we state next provide a statistical light on this process of recurrence.

Theorem 3 (von Neumann Ergodic Theorem)**.** *Let $(M, \mathcal{B}, \mu)$ be a measure space, $\varphi_t\colon M \to M$ be a measure preserving flow and $f \in L^2(\mu)$. Then*

$$\lim_{T\to+\infty} \left\| \frac{1}{T}\int_0^T f\circ\varphi_t \, dt - P_\varphi(f) \right\|_2 = 0,$$

where P_φ is the projection of f to the subspace of invariant functions by φ_t.

Proof. A proof of this von Neumann ergodic theorem can be found in p. 164 of [FH19] in Theorem 3.2.4. □

Since von Neumann ergodic theorem is a statement about ergodic properties of L^2-maps, we use this opportunity to state and prove a lemma about L^2-dynamical systems that will be important in Chapter 5:

Lemma 3. *Let (X, d) be a compact metric space, $f\colon X \to X$ an homeomorphism and μ a probability on X that is invariant under f. Also, let $\psi \in L^2(\mu)$ and $(g_k)_k$ a sequence of functions also in $L^2(\mu)$, such that*

- $g_k \underset{L^2}{\longrightarrow} \psi$;
- $g_k \circ f = g_k$ *for all* $k \in \mathbb{N}$.

Then for $\mu-$almost everywhere we have $\psi \circ f = \psi$. In other words: the L^2-limit of $f-$invariant functions is still $f-$invariant.

Proof. First we notice that $g_k \circ f \underset{L^2}{\longrightarrow} \psi \circ f$. Indeed, since μ is $f-$invariant and since $g_k \underset{L^2}{\longrightarrow} \psi$, we have:

$$\begin{aligned}
\|\psi \circ f - g_k \circ f\|_2^2 &= \int_X |\psi \circ f - g_k \circ f|^2 \, d\mu \\
&= \int_X |\psi - g_k|^2 \circ f \, d\mu \\
&= \int_X |\psi - g_k|^2 \, d\mu = \|\psi - g_k\|_2^2 \xrightarrow[k \to +\infty]{} 0.
\end{aligned}$$

So, given $\varepsilon > 0$ we have, for sufficiently large $k \in \mathbb{N}$,

$$\begin{aligned}
\|\psi \circ f - \psi\|_2 &\leq \|\psi \circ f - g_k \circ f\|_2 + \|g_k \circ f - g_k\|_2 + \|g_k - \psi\|_2 \\
&= \|\psi \circ f - g_k \circ f\|_2 + 0 + \|g_k - \psi\|_2 < 2\varepsilon,
\end{aligned}$$

because g_k is $f-$invariant for every $k \in \mathbb{N}$.

This shows that $\|\psi \circ f - \psi\|_2 = 0$. Then, $(\psi \circ f)(x) = \psi(x)$ for $\mu-$a.e. $x \in X$ or, equivalently, that ψ is $f-$invariant. □

Theorem 4 (Birkhoff Ergodic Theorem for flows)**.** *Let $(M, \mathcal{B}, \mu)$ be a probability space, $\varphi_t\colon M \to M$ a $\mu-$preserving flow on M, and $f \in L^1(\mu)$. Then,*

$$\widetilde{f}(x) = \lim_{t \to +\infty} \frac{1}{t} \int_0^t f \circ \varphi_s(x) \, ds,$$

for $\mu-$almost every $x \in M$.

Proof. For a proof of Birkhoff Ergodic Theorem for flows see Theorem 3.2.17, p.169, in [FH19]. □

Unique ergodicity

In this subsection we define the concept of unique ergodicity and give examples and conditions related to it. The notion of uniquely ergodic systems will be the theme of the main theorem presented in this text.

Definition 12. *We say a continuous flow $\varphi\colon \mathbb{R} \times M \to M$ on a metrizable compact space is **uniquely ergodic** if it has exactly one invariant Borel probability measure.*

As in the case of topological dynamics, the linear flow φ_t on $\mathbb{T}^n$ provides several scenarios for unique ergodicity. Indeed, if $\theta = (\theta_1, \ldots, \theta_n)$ has rational components, then $\varphi_t(x) = [x + t\theta]$ is periodic and then the Dirac measure supported on the periodic orbit $\mathcal{O}$, i.e.,

$$\delta_{\mathcal{O}}(x) = \begin{cases} 1, & \text{if } x \in \mathcal{O} \\ 0, & \text{if } x \notin \mathcal{O} \end{cases},$$

is an invariant measure distinct from Leb on $\mathbb{T}^n$. Hence, in this case, φ_t is **not** uniquely ergodic.

However, if the components of θ are rationally independent, we obtain unique ergodicity for the linear flow $\varphi_t(x) = [x + t\theta]$. To see that, we first remember a very important theorem about existence of measures invariant under translations on the abelian group $(\mathbb{T}^n, +)$:

Theorem (Haar)**.** *About the group* $(\mathbb{T}^n, +)$*, it holds:*

(i) *There exists some Borel measure* μ_G *on* G *that is invariant under all right-translations, finite on compact sets and positive on open sets;*

(ii) *If* η *is a measure invariant under all right-translations and finite on compact sets then* $\eta = c\mu_G$ *for some* $c > 0$.

Proof. For a proof of this theorem for a more general case where instead of $(\mathbb{T}^n, +)$ we have a Lie group $(G, \cdot)$, see Theorem 6.3.4 on p.165 of [VO16]. $\square$

An application of the previous theorem for the $\mathbb{T}^n$ is the following.

Corollary 3. *If* η *is a probability measure invariant under all right-translations and positive on open sets on* $\mathbb{T}^n$ *then* $\eta = c\,\mathrm{Leb}$ *for some* $c > 0$*. Moreover, since* η *is a probability,* $\eta(\mathbb{T}^n) = 1$ *and then* $\eta = \mathrm{Leb}$.

Now, we show unique ergodicity for $\varphi_t(x) = [x + t\theta]$ whenever the components of θ are rationally independent. In this case, Proposition 4 implies φ_t is minimal and we have the following:

Proposition 5. *If the linear flow* $\varphi_t(x) = [x + t\theta]$ *is minimal, then it is uniquely ergodic.*

Proof. For this proof we follow [VO16] and show that if μ is a probability measure invariant by φ_t, then $\mu = \mathrm{Leb}$.

So let μ be a probability measure invariant by φ_t and $x_0 \in \mathbb{T}^n$ be fixed.

Since μ is φ_t-invariant, for every $t \in \mathbb{R}$ and every continuous function $f\colon \mathbb{T}^n \to \mathbb{R}$ we have

$$\int_{\mathbb{T}^n} f(x)\, d\mu(x) = \int_{\mathbb{T}^n} f(x + t\theta)\, d\mu(x).$$

Moreover, since φ_t is minimal, every orbit is dense. Hence, there is a sequence $(t_n)_n$ with $\lim_n t_n \to +\infty$ such that

$$\lim_{n\to+\infty} \varphi_{t_n}(0) = \lim_{n\to+\infty} t_n \cdot \theta = x_0.$$

Since $\mathbb{T}^n$ is compact, for every $\varepsilon > 0$, there is a $\delta > 0$ such that for all $x, y \in \mathbb{T}^n$ with $\|x - y\| < \delta$, we have $|f(x) - f(y)| < \varepsilon$. So, if n is sufficiently large,

$$\|(x + t_n \cdot \theta) - (x + x_0)\| = \|t_n \cdot \theta - x_0\| < \delta,$$

for all $x \in \mathbb{T}^n$. In particular, $|f(x + t_n \cdot \theta) - f(x + x_0)| < \varepsilon$ for all $x \in \mathbb{T}^n$ and then,

$$\left|\int_{\mathbb{T}^n} (f(x) - f(x + x_0))\; d\mu\right| = \left|\int_{\mathbb{T}^n} (f(x + t_n \cdot \theta) - f(x + x_0))\; d\mu\right| < \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, it follows that

$$\int_{\mathbb{T}^n} f(x)\, d\mu(x) = \int_{\mathbb{T}^n} f(x + x_0)\, d\mu(x)$$

for all $x_0 \in \mathbb{T}^n$ and all continuous function $f\colon \mathbb{T}^n \to \mathbb{R}$. In other words, we have shown that μ is invariant by every right-translation on $\mathbb{T}^n$. From Corollary 3, $\mu = \mathrm{Leb}$. $\square$

Whenever we know the flow is uniquely ergodic, then we get more information on the limit function on Theorem 4:

Proposition 6. *If $\varphi\colon \mathbb{R} \times M \to M$ is uniquely ergodic then, for every continuous function $f \in C^0(M)$, the time averages*

$$\frac{1}{T}\int_0^T f \circ \varphi_t(x)\, dt$$

converge uniformly to a constant.

Proof. For a proof, see Proposition 3.3.33, at p. 177 in [FH19]. □

On the other hand, we could ask if the reciprocal holds, i.e., if the uniform convergence of the time averages to a constant implies unique ergodicity. The answer is **yes** as we will see in the next proposition.

Before, we define a notation that will be extremely useful, mainly in Chapter 5: given a flow $\varphi_s\colon M \to M$ and a continuous function $f\colon M \to \mathbb{R}$, we write the Birkhoff sum as

$$S_t(f) = \int_0^t f \circ \varphi_s\, ds,$$

for $t \in \mathbb{R}$.

Proposition 7. *Let $\varphi_s\colon M \to M$ be a continuous flow on a compact metric space M. If, for every sequence $(t_k)_k$ with $t_k \to +\infty$ such that the uniform limit*

$$\lim_{k\to+\infty} \frac{1}{t_k} S_{t_k}(f)$$

exists, the limit function is constant, then φ_s is uniquely ergodic.

Before proving Proposition 7 we need three lemmas:

Lemma 4. *Let (X, d) be a compact metric space and $\{x_t\}_{t\in\mathbb{R}}$ a family of points $x_t \in X$. Suppose there exists $x \in X$ such that, for every sequence $(t_k)_k$ with $t_k \to +\infty$, there exists a subsequence $(t_{k_j})_j$ such that $\lim_j x_{t_{k_j}} = x$. Then,*

$$\lim_{t\to+\infty} x_t = x.$$

Proof. We are going to prove that for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $d(x_t, x) < \varepsilon$ for every $t \geq N$. To do so, suppose it is not true, i.e., that there exists $\varepsilon_0 > 0$ such that for every $n \in \mathbb{N}$ there is some $t_n \geq n$ with $d(x_{t_n}, x) \geq \varepsilon_0$.

The sequence $(t_n)_n$ defined above is such that $t_n \to +\infty$. However, doesn't exist a subsequence $(x_{t_{n_j}})_j$ of $(x_{t_n})_n$ with $x_{t_{n_j}} \to x$ as $j \to +\infty$. This is a contradiction. □

Lemma 5. *A system (f, μ), where $f\colon X \to X$ is an homeomorphism on a compact metric space and μ invariant probability measure, is uniquely ergodic if, and only if, the sequence*

$$\left(\frac{1}{n}\sum_{j=0}^{n-1} \varphi \circ f^j\right)_n$$

converges uniformly to a constant function, for all $\varphi \in C^0(X)$.

Proof. This is a standard proof in ergodic theory and can be found in [VO16] (Proposition 6.1.1, p. 158). $\square$

Lemma 6. *Let $(t_k)_k$ be a sequence with $t_k \to +\infty$. If the Birkhoff averages*

$$\left(\frac{1}{t_k} S_{t_k}(f)\right)_k$$

converges uniformly to a function $\overline{f}$, then $\int_X \overline{f}\, d\mu = \int_X f\, d\mu$.

Proof. From Birkhoff's Ergodic Theorem we know that

$$\tilde{f}(x) := \lim_{t\to+\infty} \frac{1}{t} S_t(f)(x)$$

exists for $\mu-$a.e. $x \in M$. Since we are supposing that $\frac{1}{t_k} S_{t_k}(f)$ converges uniformly to $\overline{f}$, we conclude that $\tilde{f}(x) = \overline{f}(x)$ for $\mu-$a.e. $x \in M$.

Birkhoff's theorem also guarantees that

$$\int_X \tilde{f}\, d\mu = \int_X f\, d\mu$$

and hence $\int_X \overline{f}\, d\mu = \int_X \tilde{f}\, d\mu = \int_X f\, d\mu$, as claim. $\square$

We are finally ready to prove Proposition 7:

Proposition. *Let $\varphi_s \colon M \to M$ be a continuous flow on a compact metric space M. If, for every sequence $(t_k)_k$ with $t_k \to +\infty$ such that the uniform limit*

$$\lim_{k\to+\infty} \frac{1}{t_k} S_{t_k}(f)$$

exists, the limit function is constant, then φ_s is uniquely ergodic.

Proof. Let $(t_k)_k$ be a sequence such that $t_k \to +\infty$ and the uniform limit

$$\lim_{k\to+\infty} \frac{1}{t_k} S_{t_k}(f)$$

exists.

By hypothesis, the limit $\psi = \lim_{k\to+\infty} \frac{1}{t_k} S_{t_k}(f)$ must be a constant function. Now, Lemma 6 implies $\int_X \psi\, d\mu = \int_X f\, d\mu$ and, since ψ is constant and $\mu(X) = 1$ we have:

$$\psi = \psi \cdot \mu(X) = \psi \cdot \int_X 1\, d\mu = \int_X \psi\, d\mu,$$

which gives

$$\lim_{k\to+\infty} \frac{1}{t_k} S_{t_k}(f) \xrightarrow{unif} \int_X f\, d\mu.$$

Now, by compactness of $\overline{\mathcal{S}}$ on the uniform topology, which is given by Lemma 18, we know that for every sequence $(t_k)_k$ with $t_k \to +\infty$ there exists a such limit ψ for $\left(\frac{1}{t_k} S_{t_k}(f)\right)_k$. By hypothesis, this limit ψ is constant and from what we have just seen, $\psi = \int_X f\, d\mu$.

From Lemma 4,

$$\lim_{t\to+\infty} \frac{1}{t} S_t(f) \xrightarrow{unif} \int_X f\, d\mu.$$

Finally, Lemma 5 implies φ_s to be uniquely ergodic. $\square$

CHAPTER 3

Anosov and geodesic flows

In this chapter we present the main characters of the text. On Section 3.1 we give the definition, highlight properties, and give the first example of an Anosov flow. On Section 3.2 we introduce the geodesic flow and discuss several of its properties. For example, we notice it is a Hamiltonian flow, observe that in negative curvature it is an example of an Anosov flow that is never a suspension flow. Finally, we observe that in the case of negatively curved surfaces, the geodesic flow has another flow to which it is related: the horocycle flow.

3.1 Anosov Flows

This master thesis is about a very special class of flows, the so-called Anosov flows. There are still many unknown facts about Anosov flows, what makes them a current topic on mathematical research. We here exhibit some introductory facts about them.

Definition 13 (Anosov flow)**.** *A flow $\varphi_t\colon M \to M$, $t \in \mathbb{R}$, is called an* ***Anosov flow*** *if there exists a φ_t-invariant decomposition*

$$TM = E^u \oplus E^c \oplus E^s,$$

i.e., each subspace E^u, E^c and E^s, is preserved by $d\varphi_t$, E^c is the space generated by the vector field associated to the flow, and the vectors on E^u are exponentially expanded by $d\varphi_t$ and the vectors on E^s are exponentially contracted by $d\varphi_t$. In other words, there exists $\lambda, \mu \in \mathbb{R}$, with, $0 < \lambda < 1 < \mu$, such that:

$$\|d\varphi_t(v)\| \geq \mu^t \|v\|, \text{ for all } t \in \mathbb{R} \text{ and } v \in E^u,$$

and

$$\|d\varphi_t(v)\| \leq \lambda^t \|v\|, \text{ for all } t \in \mathbb{R} \text{ and } v \in E^s.$$

The spaces E^u and E^s are called unstable and stable spaces, respectively.

The first example of Anosov flow we present here is the suspension flow of an Anosov diffeomorphism. A diffeomorphism is called Anosov if there exists a $f-$invariant splitting

$$TM = E^u \oplus E^s,$$

i.e., each subspace E^u and E^s, is preserved by df, and the vectors on E^u are exponentially expanded by df and the vectors on E^s are exponentially contracted by df. In other words, there exists $\lambda, \mu \in \mathbb{R}$, with,

$0 < \lambda < 1 < \mu$, such that:

$$\|df^n(v)\| \geq \mu^n \|v\|, \text{ for all } v \in E^u \text{ and } n \geq 0,$$

and

$$\|df^n(v)\| \leq \lambda^n \|v\|, \text{ for all } v \in E^s \text{ and } n \geq 0.$$

As in the case of flows, the spaces E^u and E^s are called unstable and stable spaces, respectively.

A well-known example of Anosov diffeomorphism is the Anosov Cat Map, that is defined as follows: let $A\colon \mathbb{R}^2 \to \mathbb{R}^2$ be the linear map defined by

$$Av = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \cdot v.$$

Since $A(\mathbb{Z}^2) \subseteq \mathbb{Z}^2$ and $\det A = 1$, we have an induced map $f_A\colon \mathbb{T}^2 \to \mathbb{T}^2$ from $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ to itself defined by $f_A([v]) = [Av]$, where $[v] = \{w \in \mathbb{R}^2 \mid w - v \in \mathbb{Z}^2\} \in \mathbb{T}^2$ and the definition of f_A does not depend on the element w in the class $[v]$. Now, this map $f_A\colon \mathbb{T}^2 \to \mathbb{T}^2$ is an Anosov diffeomorphism on the torus $\mathbb{T}^2$, called ***Anosov's Cat Map***, and has very interesting dynamical properties such as topological minimality and the fact that the set of periodic points of f_A is dense on $\mathbb{T}^2$. More details on Anosov diffeomorphisms as well as an argument on why f_A is an example of them is discussed in Appendix A.

Once in possession of an Anosov diffeomorphism, there is a somewhat canonical Anosov flow associated to it, that is obtained via a general construction from the qualitative theory of ODE's. Let $X \in \mathfrak{X}^r(M)$ be a vector field on M and $\Sigma \subset M$ be a compact codimension one submanifold of M. We call Σ a **global transverse section of** X if X is transverse to Σ and every positive orbit of X through each point $p \in M$ intersects Σ again in the future.

Then, if Σ is a global transverse section of a vector field $X \in \mathfrak{X}^r(M)$, the flow associated to X induces a diffeomorphism $P\colon \Sigma \to \Sigma$ that associates, for each point $x \in \Sigma$ the first point $P(x)$ such that the positive orbit $\mathcal{O}^+_\varphi(x) = \{\varphi_t(x) \mid t \geq 0\}$ intersects the section Σ again, i.e., for $t > 0$. The map P is called the ***Ponincaré map associated to*** Σ.

A very important feature of the Poincaré map is that the orbit structure of the flow φ_t associated to X is determined by its Poincaré map (and vice-versa). For example, a point $x \in \Sigma$ is a periodic point of P if and only if $\mathcal{O}_\varphi(x)$ is closed.

Suspension flow

Now we go on the other direction and from a diffeomorphism construct a flow: on this subsection we present a construction that allow us to, for each C^r−diffeomorphism $f\colon M \to M$, find a flow φ_t, called the ***suspension flow of*** f, such that f is conjugated to the Poincaré map P of φ_t.

Let M be a compact Riemannian manifold and $f\colon M \to M$ be a C^r diffeomorphism, $(r \geq 1)$.

On $M \times \mathbb{R}$, define the following equivalence relation:

$$(p, s) \sim (q, t) \iff s - t = n \in \mathbb{Z} \text{ and } q = f^n(p).$$

Let $\widetilde{M}$ be the quotient space $M \times \mathbb{R}/\sim$ and let $\pi\colon M \times \mathbb{R} \to \widetilde{M}$ be the projection map, i.e., for each pair (p, s), the image $\pi(p, s)$ is the equivalence class of (p, s).

For each $s_0 \in \mathbb{R}$ the restriction of π to $M \times (s_0, s_0 + 1)$ is an one-to-one correspondence between $M \times (s_0, s_0 + 1)$ and $\widetilde{M} - \pi(M \times s_0)$.

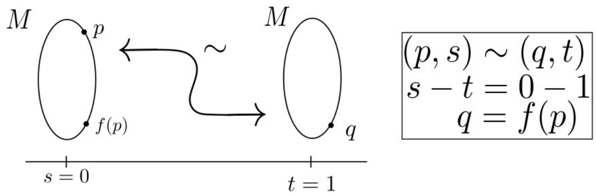

Figure 3.1: The relation $\sim$ on $M \times \mathbb{R}$.

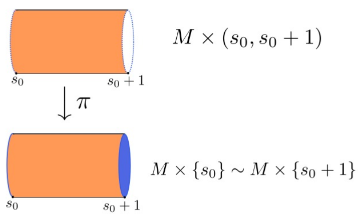

Figure 3.2: Restricted to $M \times (s_0, s_0+1)$, π is $1-1$.

To make $\widetilde{M}$ a topological space, we set $\widetilde{M}$ with the topology induced by π, i.e., $\widetilde{A} \subseteq \widetilde{M}$ is open if and only if $\pi^{-1}(\widetilde{A}) \subseteq M \times \mathbb{R}$ is open. Moreover, we want to show that $\widetilde{M}$ is not only a topological space but a smooth manifold.

To do so, we define an atlas for $\widetilde{M}$. First, let $x_i \colon U_i \to U_0 \subseteq \mathbb{R}^n$ be local charts on M for $i = 1, \ldots, k$, with $\bigcup_{i=1}^k U_i = M$. Now let $\widetilde{U}_i$ and $\widetilde{V}_i$ be two families of open sets on $\widetilde{M}$, each defined as:

$$\widetilde{U}_i = \pi\left(U_i \times \left(-\frac{1}{2}, \frac{1}{2}\right)\right) \text{ and } \widetilde{V}_i = \pi\left(U_i \times \left(\frac{1}{4}, \frac{5}{4}\right)\right),$$

for $i = 1, \ldots, k$. Each $\widetilde{U}_i$ and $\widetilde{V}_i$ is open on $\widetilde{M}$. Indeed, both

$$\pi^{-1}(\widetilde{U}_i) = \pi^{-1}\left(\pi\left(U_i \times \left(-\frac{1}{2}, \frac{1}{2}\right)\right)\right) = \bigcup_{n\in\mathbb{Z}}\left[U_i \times \left(-\frac{1}{2}+n, \frac{1}{2}+n\right)\right]$$

and

$$\pi^{-1}(\widetilde{V}_i) = \pi^{-1}\left(\pi\left(V_i \times \left(\frac{1}{4}, \frac{5}{4}\right)\right)\right) = \bigcup_{n\in\mathbb{Z}}\left[U_i \times \left(\frac{1}{4}+n, \frac{5}{4}+n\right)\right]$$

are open in $M \times \mathbb{R}$, for every $i = 1, \ldots, k$.

Now, for each $i = 1, \ldots, k$, define $\widetilde{x}_i \colon \widetilde{U}_i \to U_0 \times \left(-\frac{1}{2}, \frac{1}{2}\right)$ by $\widetilde{x}_i(\pi(p, s)) = (x_i(p), s)$ and $\widetilde{y}_i \colon \widetilde{V}_i \to U_0 \times \left(\frac{1}{4}, \frac{5}{4}\right)$ by $\widetilde{y}_i(\pi(p, s)) = (x_i(p), s)$.

Observe that both $\widetilde{x}_i$ and $\widetilde{y}_i$ are well-defined since, in each open set $U_i \times \left(-\frac{1}{2}, \frac{1}{2}\right)$ and $U_i \times \left(\frac{1}{4}, \frac{5}{4}\right)$ there exists a unique element of each class (p, s). Moreover, since the restriction of π to $U_i \times \left(-\frac{1}{2}, \frac{1}{2}\right)$ is one-to-one and the same holds for the restriction of π to $U_i \times \left(\frac{1}{4}, \frac{5}{4}\right)$, both $\widetilde{x}_i$ and $\widetilde{y}_i$ are homeomorphisms. We claim that $\mathcal{A} = \{(\widetilde{x}_i, \widetilde{U}_i), (\widetilde{y}_i, \widetilde{V}_i) \mid i = 1, \ldots, k\}$ is a C^r-atlas on $\widetilde{M}$. Indeed, by making a small notation abuse and confusing (p, s) with its equivalence class, we have:

$$\begin{aligned}
\widetilde{x}_i \circ \widetilde{x}_j^{-1}(u, s) &= \widetilde{x}_i(\widetilde{x}_j^{-1}(u, s)) = \widetilde{x}_i(\widetilde{x}_j^{-1}(u), s) = (x_i \circ x_j^{-1}(u), s),\\
\widetilde{y}_i \circ \widetilde{y}_j^{-1}(u, s) &= \widetilde{y}_i(\widetilde{y}_j^{-1}(u, s)) = \widetilde{y}_i(\widetilde{y}_j^{-1}(u), s) = (y_i \circ y_j^{-1}(u), s).
\end{aligned}$$

This proves that $\widetilde{x}_i \circ \widetilde{x}_j^{-1}$ and $\widetilde{y}_i \circ \widetilde{y}_j^{-1}$ are C^r-diffeomorphisms. For the last case, i.e., $\widetilde{x}_i \circ \widetilde{y}_j^{-1}(u, s)$, we need to be a little more careful: since $\widetilde{y}_j^{-1}(u, s) \in \widetilde{V}_i$ we pick $(f(x_j^{-1}(u)), s-1) \in \widetilde{U}_i$ that satisfies $(f(x_j^{-1}(u)), s-1) \sim (x_j^{-1}(u), s)$. We then obtain:

$$\widetilde{x}_i \circ \widetilde{y}_j^{-1}(u, s) = \widetilde{x}_i(\widetilde{y}_j^{-1}(u, s)) = \widetilde{x}_i(f(x_j^{-1}(u)), s-1) = (x_i(f(x_j^{-1}(u)), s-1),$$

proving that $\widetilde{x}_i \circ \widetilde{y}_j^{-1}(u, s)$ is also a C^r-diffeomorphism. This shows that $\mathcal{A}$ is a C^r atlas.

Actually, we can consider $\widetilde{M}$ as a C^∞ manifold since, from Whitney's Embedding Theorem, there is a smooth manifold structure on $\widetilde{M}$ such that the maps $\widetilde{x}_i$ and $\widetilde{y}_i$ are C^r diffeomorphisms.[1]

[1]For instance, see Theorem 0.19 on p. 9 of [PD12].

Observe that π is a C^r-local diffeomorphism since, restricted to $U_0 \times \left(-\frac{1}{2}, \frac{1}{2}\right)$, the composition $\widetilde{x}_i \circ \pi \circ (x_i^{-1} \times \mathbf{Id})$ is the identity, and also, restricted to $U_0 \times \left(\frac{1}{4}, \frac{5}{4}\right)$ the map $\widetilde{y}_i \circ \pi \circ (x_i^{-1} \times \mathbf{Id})$. Now, let $\frac{\partial}{\partial t}$ be the unit vector on $M \times \mathbb{R}$ whose orbits are the lines $\{p\} \times \mathbb{R}$, for $p \in M$. Define,

$$X(\pi(p,s)) = d\pi_{(p,s)} \cdot \frac{\partial \pi}{\partial t}(p,t).$$

From the definition of the relation $\sim$, it is straightforward to see that $X(\pi(p,s)) = X(\pi(f(p), s-1))$ and that X is a C^{r-1} vector field on $\widetilde{M}$.

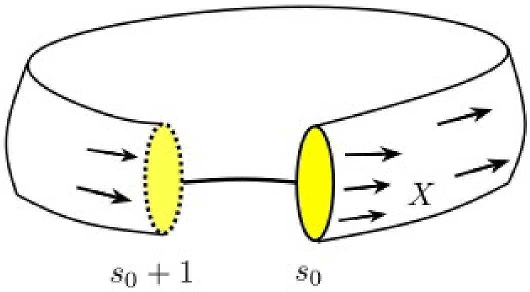

Figure 3.3: The vector field X on $\widetilde{M}$.

If we call $\widetilde{\Sigma} \subseteq \widetilde{M}$ the projection of $M \times \{0\}$ by π, i.e., $\widetilde{\Sigma} = \pi(M \times \{0\})$, the vector field X is transverse to $\widetilde{\Sigma}$ and its orbit through $\widetilde{p} = \pi(p,t)$ is $\pi(\{p\}, \mathbb{R})$.

It can be shown that the positive orbit of X through a point $\widetilde{p} = \pi(p, 0) \in \widetilde{\Sigma}$ return to $\widetilde{\Sigma}$ again for the first time at the point $\widetilde{q} = \pi(p,1) = \pi(f(p), 0)$. Moreover, the Poincaré map associated to the transverse section $\widetilde{\Sigma}$ is $\widetilde{f}\colon \widetilde{\Sigma} \to \widetilde{\Sigma}$ defined by $\widetilde{f}(\pi(p,0)) = \pi(f(p), 0)$. Hence, the map $h\colon M \to \widetilde{\Sigma}$ defined by $h(p) = \pi(p, 0)$ is a C^r diffeomorphism which conjugates f and $\widetilde{f}$, i.e., $\widetilde{f} \circ h = h \circ f$.

A comment on suspension flows that will be extremely important for this text is that, even if the dynamics before the suspension is topologically mixing, the suspension doesn't need to be:

Proposition 8. *If φ_t is the suspension flow of a diffeomorphism $f\colon M \to M$ then it fails to be topologically mixing.*

Proof. We follow the ideas in [Day20]. As before, write the suspension flow $\varphi_t\colon M \to M$, where $\widetilde{M} = M \times \mathbb{R}/\sim$ and, for small values of t, $\varphi_t(x,s) = (x, s+t)$. Consider $U, V \subset M$ open sets and fix $U' = U \times (0, 1/4)$ and $V' = V \times (0, 1/4)$ open sets in $\widetilde{M}$.

Notice that, for $a \in (0, 1/4)$, we can identify $U \times \{a\} \subset U'$ with $f(U) \times \{0\}$ by taking $t = 1 - a$. Indeed, we just have to notice that, for all $x \in U$:

$$\varphi_t(x,a) = (x, t+a) = (x, 1) = (f(x), 0).$$

This implies that, for each $x \in U$, $\operatorname{diam} \varphi_t\,(x \times (0, 1/4)) = 1/4$.

Now, suppose there exists T such that $\varphi_T(U') \cap V' \neq \emptyset$. Since $\operatorname{diam} u \times (0,1/4) = 1/4$ for all $u \in U$ and $\operatorname{diam} v \times (0, 1/4) = 1/4$ for all $v \in V$, the observation on the last paragraph shows that

$$\varphi_{T+1/4}(U') \cap V' = \emptyset,$$

proving that φ_t cannot be topologically mixing. $\square$

In fact, there is a deeper result in that sense, that is proven in [Day20]. To present it here, we need a generalization on the construction of the suspension flow. The next definition follows the one given in [BS02].

Definition. *Given a map $f\colon X \to X$ and a function $h\colon X \to \mathbb{R}^+$ bounded away from 0, consider the quotient space*

$$M_h = \{(x,t) \in X \times \mathbb{R}^+ \mid 0 \leq t \leq h(x)\}/\sim,$$

*where $\sim$ is the equivalence relation $(x, h(x)) \sim (f(x), 0)$. In this setting, the **suspension** of f with **height function** $h(x)$ is the flow $\varphi_t(x,s) = (f^n(x), s')$ where n and s' are given by:*

$$\sum_{j=0}^{n-1} h(f^j(x)) + s' = t + s,$$

and

$$0 \leq s' \leq h(f^n(x)).$$

This definition generalizes the previous construction, since there we have taken the height function h to be $h(x) \equiv 1$. Together with the smooth structure of M and the regularity of f on the previous case, this allowed us to obtain several properties for the quotient space $\widetilde{M}$ and the suspension flow φ_t. To obtain the topologically mixing property for φ_t we need to make hypothesis on the roof function h:

Theorem. *Let X be a compact metric space and $f\colon X \to X$ be a topologically mixing homeomorphism with dense periodic points. A suspension flow over X is topologically mixing if and only if the height function $h\colon X \to \mathbb{R}$ is not cohomologous to a constant k, i.e., if there isn't a continuous function $g\colon X \to \mathbb{R}$ such that $h(x) - k = g(f(x)) - g(x)$ for all $x \in X$.*

The last paragraphs explained some of the difficulties for the suspension flow to be topologically mixing even if the base dynamics f is. However, if we ask for the diffeomorphism $f\colon M \to M$ to be Anosov, then it is a somewhat stronger assumption: the suspension flow will be an Anosov flow.

Proposition 9. *The suspension flow $\varphi_t\colon \widetilde{M} \to \widetilde{M}$ of an Anosov diffeomorphism $f\colon M \to M$, is an Anosov flow.*

Proof. To see this, fix a point $(x,\theta) \in \widetilde{M}$. We need to show that there are contracting and expanding subspaces Es and E^u, respectively, in $T_{(x,\theta)}\widetilde{M}$ such that

$$T_{(x,\theta)}\widetilde{M} = E^s \oplus \langle X \rangle \oplus E^u.$$

This occur since, at each point $x \in M$, such spaces E^s and E^u are defined in T_xM and the vector field X defined above is transverse to T_xM in $T_{(x,\theta)}M$.

Moreover, this decomposition is invariant by $d\varphi_t\colon T_{(x,\theta)}\widetilde{M} \to T_{\varphi_t(x,\theta)}\widetilde{M}$. Indeed, if $0 \leq t+s < 1$, we have that $\varphi_t(x,\theta) = (x, t+\theta)$ and $T_{\varphi_t(x,\theta)}\widetilde{M} = E^s(x) \oplus \langle X \rangle \oplus E^u(x)$. If $t+s = 1$, then $\varphi_t(x,\theta) = (x, t+\theta) = (f(x), 0)$, and hence $d\varphi_t(x,\theta) = (df_x, X)$ and

$$T_{\varphi_t(x,\theta)}\widetilde{M} = T_{(x,t+\theta)}\widetilde{M} = T_{(f(x),0)}\widetilde{M}.$$

Now, since $f\colon M \to M$ is an Anosov diffeomorphism, the decomposition $E^s(x) \oplus E^u(x)$ is invariant under df, the decomposition $E^s(x,\theta) \oplus \langle X(x,\theta)\rangle \oplus E^u(x,\theta)$ is invariant under $d\varphi_t$, i.e.,

$$d\varphi_t\left(E^s(x,\theta) \oplus \langle X\rangle \oplus E^u(x,\theta)\right) = E^s(\varphi_t(x,\theta)) \oplus \langle X(\varphi_t(x,\theta))\rangle \oplus E^u(\varphi_t(x,\theta)).$$

Finally, for each section $\Sigma_t = M \times \{t\}$ of the suspension manifold, we have that $\varphi_n\big|_{\Sigma_t} = f^n$. Since f is Anosov and the uniform expansion and contraction with respect to each space E^u and E^s occur in a regular interval gap, we must have uniform expansion and contraction for φ_t on E^u and E^s as well. This proves the flow φ_t is Anosov on $\widetilde{M}$.

$\square$

After Proposition 9 we are now able to give the first concrete example of Anosov flow:

Example 9 (The suspension flow of the Cat Map)**.** *When the Anosov diffeomorphism is the Cat Map presented above, the construction of the suspension flow, allows us to find a vector field X on a 3-manifold ($\mathbb{T}^2 \times \mathbb{R}/\sim$), whose associated flow is Anosov, and such that its Poincaré map is conjugated to the Cat Map, and hence has all its interesting dynamical properties.*

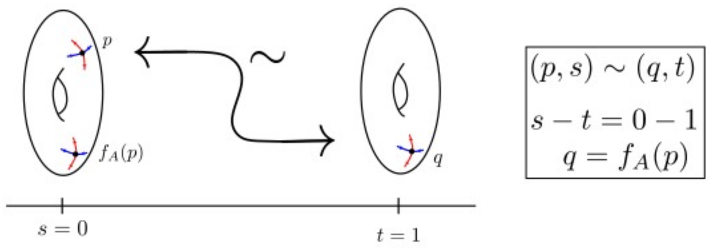

Figure 3.4: Suspension manifold for the Cat Map.

Stable manifold theorem for Anosov flows

Just as in the case of Anosov diffeomorphisms, Anosov flows have important and beautiful theorems that make solid the theory and give tools for us to seek results. The first major result we present here is the Stable Manifold Theorem for flows:

Theorem 5 (Stable Manifold Theorem)**.** *Let $g_t\colon M \to M$ be a C^r ($r \geq 1$) Anosov flow on M. Fix $t > 0$ and let $\lambda, \mu \in \mathbb{R}$, with, $\lambda < 1 < \mu$, be as in definition of Anosov flow, i.e.,*

$$\|dg_t(v)\| \geq \mu^t \|v\|, \textit{ for all } t \in \mathbb{R} \textit{ and } v \in E^u,$$

and

$$\|dg_t(v)\| \leq \lambda^t \|v\|, \textit{ for all } t \in \mathbb{R} \textit{ and } v \in E^s.$$

Then, for each $x \in M$ there is a pair of embedded C^r-discs, $W^{ss}_{loc}(x)$ and $W^{uu}_{loc}(x)$ (the local strong stable manifold and the local strong unstable manifold of x, respectively) such that:

(i) $T_x(W^{ss}_{loc}(x)) = E^s(x)$ *and* $T_x(W^{uu}_{loc}(x)) = E^u(x)$;

(ii) $g_t(W^{ss}_{loc}(x)) \subset W^{ss}_{loc}(g_t(x))$ *and* $g_{-t}(W^{uu}_{loc}(x)) \subset W^{uu}_{loc}(g_{-t}(x))$, *for every* $t \geq t_0 > 0$;

(iii) *for every* $\delta > 0$ *there exists* $C(\delta)$ *such that*

$$d(g_t(x), g_t(y)) < C(\delta)(\lambda + \delta)^t d(x, y), \text{ for } y \in W^{ss}_{loc}(x) \text{ and } t > 0$$
$$d(g_{-t}(x), g_{-t}(y)) < C(\delta)(\mu - \delta)^{-t} d(x, y), \text{ for } y \in W^{uu}_{loc}(x) \text{ and } t > 0;$$

(iv) *there exists a continuous family* U_x *of neighborhoods of* $x \in M$ *such that:*

$$W^{ss}_{loc}(x) = \left\{ y \in M \mid g_t(y) \in U_{g_t(x)}, \text{ for all } t > 0 \text{ and } \lim_{t\to+\infty} d(g_t(y), g_t(x)) = 0 \right\},$$
$$W^{uu}_{loc}(x) = \left\{ y \in M \mid g_{-t}(y) \in U_{g_{-t}(x)}, \text{ for all } t > 0 \text{ and } \lim_{t\to+\infty} d(g_{-t}(y), g_{-t}(x)) = 0 \right\}.$$

Proof. A proof of the stable manifold theorem for flows is presented at Theorem 17.4.3, p. 545, in [KH97]. □

Whenever it is important to specify the size of the local stable/unstable manifold, we will use the following notation (for example for the weak unstable manifold):

$$W^u_\varepsilon(x) = \{y \in M \mid d(g_{-t}(x), g_{-t}(y)) < \varepsilon, \text{ for all } t \geq 0\}.$$

A particular consequence of the Stable Manifold Theorem for Anosov flows is the existence of foliations on the manifold M that are related to the dynamics.

The Theorem implies that each tangent bundles E^u, E^s, $E^u \oplus E^c$, and $E^s \oplus E^c$, are uniquely integrable[2] and give rise to foliations tangent to it. For each bundle we have the analogue foliation as follows:

Bundle	Foliation
$E^u \oplus E^c$	$\mathcal{F}^u$
$E^s \oplus E^c$	$\mathcal{F}^s$
E^u	$\mathcal{F}^{uu}$
E^s	$\mathcal{F}^{ss}$

We call $\mathcal{F}^{uu}$ and $\mathcal{F}^{ss}$ by the unstable and stable foliations, respectively, and $\mathcal{F}^u$ and $\mathcal{F}^s$ by center-unstable and center-stable foliations, respectively. An important comment to make, is that the leafs of each foliation is of the same class of differentiability of the the flow. However, for the regularity of the foliation itself, we may only ask for it to be C^0. For more details on the regularity of the invariant foliations, see [PSW97].

The second major theorem we need to state is known as the Product Neighborhood Theorem. To do so, we first introduce some notation: let d_u, d_s, d_{uu}, d_{ss} be the metrics induced by d on the leaves of the foliations $\mathcal{F}^u, \mathcal{F}^s, \mathcal{F}^{uu}, \mathcal{F}^{ss}$, respectively. Define, for $x \in M$ and $\delta > 0$, the following sets:

$$B_\delta(x) = \{y \in M \ : \ d(x, y) < \delta\}$$
$$B^u_\delta(x) = \{y \in W^u(x) \ : \ d_u(x, y) < \delta\}$$
$$B^s_\delta(x) = \{y \in W^s(x) \ : \ d_s(x, y) < \delta\}$$
$$B^{uu}_\delta(x) = \{y \in W^{uu}(x) \ : \ d_{uu}(x, y) < \delta\}$$
$$B^{ss}_\delta(x) = \{y \in W^{ss}(x) \ : \ d_{ss}(x, y) < \delta\}$$

[2]See Definition 4.

Theorem 6. *(Product Neighborhood Theorem) There exists $\delta_0 > 0$, not depending on $x \in M$, such that for $\delta \leq \delta_0$ the functions*

$$G\colon B^s_\delta(x) \times B^{uu}_\delta(x) \to M$$
$$H\colon B^{ss}_\delta(x) \times B^u_\delta(x) \to M$$

given by

$$G(y,z) = B^s_{2\delta}(z) \cap B^{uu}_{2\delta}(y)$$
$$H(y,z) = B^{ss}_{2\delta}(z) \cap B^u_{2\delta}(y)$$

are well defined and are homeomorphisms onto its images.

This images are called product neighborhoods of x. The proof uses Theorem 5 and simple techniques from hyperbolic dynamics. It can be found in Section 6.2 of [FH19] and, more precisely, it is a consequence of Proposition 6.2.2 and Theorem 6.2.7 of the same reference.

Another very important theorem on hyperbolic dynamics that also has some geometric flavor and is related to the Stable Manifold Theorem is the $\lambda-$Lemma (also known as Inclination Lemma). We present here two versions of it (Propositions 6.1.7 and 6.1.10 at p.335 of [FH19]):

Theorem 7 (λ-lemma for fixed points)**.** *Suppose p is a hyperbolic fixed point of a smooth flow $\varphi_t\colon M \to M$ and D is a disk that intersects $W^s(p)$ transversely. Then, the sets $\varphi_t(D)$ accumulate on $W^u(p)$ in the C^1-topology, as $t \to +\infty$. Specifically, for any disk Δ in $W^u(p)$ and any $\varepsilon > 0$, there is an instant $t > 0$ and $D' \subset D$ such that $d_{C^1}(\varphi_t(D'), \Delta) < \varepsilon$.*

The second version treats more explicitly what happens at the center-unstable manifolds:

Theorem 8 (λ-lemma for flows)**.** *Suppose p is a hyperbolic periodic point for a flow $\varphi_t\colon M \to M$, of least period $T > 0$, and suppose T_pM has splitting $T_pM = E^s \oplus E^c \oplus E^u$. Let D be an embedded disk that intersects $W^s(p)$ transversely in some point $q \in W^s(p)$ such that* $\dim D = \dim E^u + 1$. *Then, for any $\varepsilon > 0$, there exists an order $N \in \mathbb{N}$ such that, for each $n \geq N$, there is an embedded disk $D_n \subseteq D$ containing q such that $\varphi_{t_n}(D_n)$ is $\varepsilon-$close to $W^u(p)$ in the C^1-topology.*

An useful way to understand the dynamics of a system is to restrict ourselves to the set of periodic points of φ_t, denoted by, $\mathrm{Per}\,(\varphi_t)$. Now, we present two theorems that corroborate the claim we've just made.

The first one is about how big (topologically) the set of periodic points may be: its closure is at least as big as the non-wondering set. Before stating it precisely, we need a definition:

Theorem 9 (Anosov Closing Lemma)**.** *Let M be a closed and connected Riemannian manifold and $\varphi_t\colon M \to M$ an Anosov flow. Every recurrent point $x \in M$ is approximated by a periodic point.*

Proof. See Corollary 18.1.8, p. 570, from [KH97]. □

Corollary 4. *The periodic points of φ_t are dense in $\Omega(\varphi)$, i.e.,* $\overline{\mathrm{Per}\,(\varphi)} = \Omega(\varphi)$.

Proof. See Corollary 5.3.22, p. 298 of [FH19]. □

The next result is about the structure of $\overline{\text{Per}(\varphi)}$. It states that this set can be broken into smaller invariant and disjoint pieces, each of which is transitive when the dynamics is restricted to it. These pieces are called basic sets for the dynamics.

Definition 14 (Basic sets)**.** *Let $\varphi_t \colon M \to M$ be an Anosov flow on a closed connect Riemannian manifold M. A closed subset $\Lambda \subset M$ is called* ***isolated*** *if there is a neighborhood V of Λ (**isolating neighborhood**) such that $\Lambda = \bigcap_{t\in\mathbb{R}} \varphi_t\left(\overline{V}\right)$. Moreover, if we also have that $\varphi_t\big|_{\Lambda}$ is transitive, the set Λ is called a* ***basic set****.*

Theorem 10 (Spectral Decomposition)**.** *For an Anosov flow $\varphi_t \colon M \to M$ on a closed connected manifold M, there exists a finite family $\Lambda_1, \dots, \Lambda_k$ of compact disjoint invariant basic sets such that*

$$\overline{\text{Per}(\varphi)} = \Lambda_1 \cup \dots \cup \Lambda_k.$$

Proof. For a proof check Theorem 5.3.37, p. 302, of [FH19]. □

Back to the problem of how big or representative the set of periodic points are, we may ask whether its closure is equal to the hole manifold. By Corollary 4, we know that, if φ_t is an Anosov flow, this question is equivalent as asking whether the non-wandering set is the whole manifold. More precisely, the following holds:

Theorem 11. *Let $\varphi_t \colon M \to M$ be an Anosov flow on a closed connected Riemannian manifold M. The following properties are equivalent:*

(i) *The spectral decomposition of φ has only one piece, which is M itself;*

(ii) $\Omega(\varphi) = M$*;*

(iii) $\overline{\text{Per}(\varphi)} = M$*;*

(iv) *φ_t is topologically transitive;*

(v) *All center-stable leafs are dense, i.e., $\overline{W^s(x)} = M$ for all $x \in M$;*

(vi) *All center-unstable leafs are dense, i.e., $\overline{W^u(x)} = M$ for all $x \in M$.*

Proof. The proof of this theorem is, in some extent, a consequence of Theorem 10. A proof can be found in p.342 of [FH19], being the proof of Theorem 6.2.10 of such text. □

It was a long standing conjecture the claim that if $\varphi_t \colon M \to M$ is Anosov flow, then $\Omega(\varphi) = M$. Unfortunately, this conjecture was disproved in [FW80], at least in the generality it was presented.

Nevertheless, there are very interesting examples for which the conjecture works, namely the suspension flow of an Anosov diffeomorphism and the geodesic flow on a manifold with constant negative curvature.

Proposition 10. *If $f \colon M \to M$ is a transitive Anosov diffeomorphism on a closed connected Riemannian manifold M, then $\varphi_t \colon \widetilde{M} \to \widetilde{M}$ its suspension flow is an Anosov flow such that $\Omega(\varphi) = M$.*

The statement that the suspension flow of an Anosov diffeomorphism is itself an Anosov flow was the content of Proposition 9. Hence, to prove this Proposition 10 we need to check that if f is transitive, its suspension flow satisfy $\Omega(\varphi) = M$. In order to do that we need a lemma:

Lemma 7. *If $f\colon M \to M$ is an transitive Anosov diffeomorphism on a closed connected Riemannian manifold M, then f is topologically mixing.*

Proof. The proof of this Lemma 7 will be done in detail in Appendix A, precisely in Theorem 26. □

Proof of Proposition 10. Now, consider two non-empty open sets $\widetilde{U}, \widetilde{V}$ on $\widetilde{M}$. We need to show that there exist some $s \in \mathbb{R}$ such that $\varphi_s(\widetilde{U}) \cap \widetilde{V} \neq \emptyset$.

By construction of the suspension manifold $\widetilde{M}$, we can suppose, without loss of generality, that $\widetilde{U} = U \times I_U$ and $\widetilde{V} = V \times I_V$, where U and V are open sets on vertical sections $M \times \{t_U\}$ and $M \times \{t_V\}$ of $\widetilde{M}$, respectively, for $t_U, t_V \in \mathbb{R}$, and I_U and I_J are small intervals.

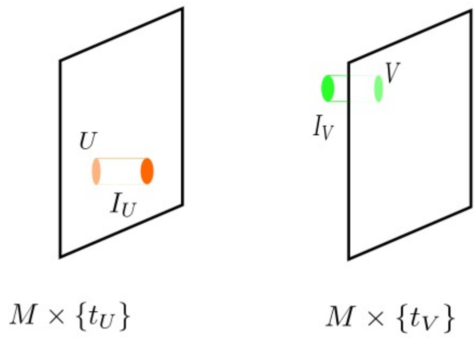

Figure 3.5: The open sets $\widetilde{U}$ and $\widetilde{V}$.

From Appendix A we know that the transitive Anosov diffeomorphism $f\colon M \to M$ is also topologically mixing. So, given $\varepsilon > 0$ there exists $N > 0$ such that, if $n \geq N$ then $f^n(U) \cap V \neq \emptyset$. In particular, since M is compact, there exists $N > 0$ such that, if $n \geq N$ then $f^n(U)$ is $\varepsilon-$dense on M.

Since φ_t is the suspension flow, it is tangent to the vector field $\frac{\partial}{\partial t}$, so by its action the interval I_V will intersect the section $\varphi_s(M \times t_U)$ for an infinite number of instants s. In particular, there is some $s > N$ such that $\varphi_s(M \times \{t_U\}) \cap \widetilde{V} \neq \emptyset$. Since we took s to be greater than N, the set $f^{\lfloor s \rfloor}(U)$ will be $\varepsilon-$dense on M. So, by first reducing the choice of $\varepsilon > 0$ if necessary, we have that $\varphi_s(\widetilde{U}) \cap \widetilde{V} \neq \emptyset$. □

Example 10. *Since the Anosov Cat Map $f_A\colon \mathbb{T}^2 \to \mathbb{T}^2$ on $\mathbb{T}^2$ is transitive, we have $\Omega(f_A) = \mathbb{T}^2$. By the previous proposition, its suspension flow φ_t also satisfy $\Omega(\varphi) = \widetilde{\mathbb{T}^2}$, where $\widetilde{\mathbb{T}^2}$ is the suspension manifold of $\mathbb{T}^2$ obtained by the suspension of f_A.*

The next example, the geodesic flow on a manifold with constant negative curvature, is the main topic of this text and we reserve the next section to introduce it.

3.2 Geodesic Flow on T^1M

Let $(M, \langle \cdot, \cdot \rangle)$ be a closed Riemannian manifold of dimension n, where $\langle \cdot, \cdot \rangle$ is a Riemannian metric on M, and let ∇ denote its Riemannian connection.

Definition 15 (Geodesic)**.** *A curve $\gamma\colon I \to M$ is called a geodesic if*

$$\nabla_{\gamma'} \gamma' = 0.$$

Given a coordinate system (U, x), with U an open set around a point $p \in M$, write X_i for the vector field $X_i = \frac{\partial}{\partial x_i}$ on U. Since $\{X_1, \ldots, X_n\}$ is a frame of $TM\big|_U$, we can write $\nabla_{X_i} X_j = \sum_k \Gamma_{ij}^k X_k$. The real valued functions Γ_{ij}^k are called the ***Christoffel symbols*** of the connection ∇ in the chart (U, x).

Proposition 11. *A curve $\gamma \colon I \to M$ is a geodesic on M if and only if, in any coordinate system,*

$$\gamma_k'' + \sum_{i,j} \Gamma_{ij}^k \gamma_i' \gamma_j' = 0, \tag{3.1}$$

for all $k = 1, \ldots, n$.

Proof. The proof of this fact follows immediately from a computation in local coordinates.

□

Given a manifold M, the tangent bundle TM of M is defined as the disjoint union of tangent spaces T_pM over all $p \in M$, i.e.,

$$TM = \bigsqcup_{p \in M} T_pM$$

and is also a manifold, now of dimension $2n$. Remember that $v \in T_pM$ if and only if there exists a smooth curve $\gamma \colon I \to M$, from some interval I around $0 \in \mathbb{R}$ to M, such that $\gamma(0) = p$ and $\gamma'(0) = v$.

Moreover, observe that $\|\gamma'(t)\|$ is constant for every $t \in I$. Indeed,

$$\begin{aligned} \frac{d}{dt}\|\gamma'\| &= \frac{d}{dt}\langle \gamma', \gamma' \rangle \\ &= 2 \cdot \langle \nabla_{\gamma'}\gamma', \gamma' \rangle \\ &= 0, \end{aligned}$$

since $\nabla_{\gamma'}\gamma' = 0$.

From the theory of ODE's, we obtain two properties: first, for each point $p \in M$ and vector $v \in T_pM$ there is a unique geodesic $\gamma \colon I \to M$ with $\gamma(0) = p$ and $\gamma'(0) = v$, defined on its maximal interval of definition I. To prove it, we must solve the system of equations (3.1). Secondly, since $\|\gamma'\|$ is constant, the maximal interval of definition is, in fact, $\mathbb{R}$, i.e., the geodesics are defined for all values $t \in \mathbb{R}$.

Notice that each vector $w \in T_qM$, where q is a point in the selected local coordinate system, can be written as $w = \sum_{k=1}^n y_k X_k$. So, the point $(q, w) \in TU$ is written as $(\gamma_1, \ldots, \gamma_n, y_1, \ldots, y_n)$ and these coordinates give a smooth structure on TM.

So, each curve $\gamma(t)$ on M determines a curve $(\gamma(t), \gamma'(t))$ on TM. If γ is a geodesic on $U \subset M$ that is written in local coordinates as $\gamma(t) = (\gamma_1(t), \ldots, \gamma_n(t))$, the curve

$$t \mapsto \big(\gamma_1(t), \ldots, \gamma_n(t), \gamma_1'(t), \ldots, \gamma_n'(t)\big)$$

on TU is a solution to the system

$$\begin{cases} \frac{dx_k}{dt} = y_k \\ \frac{dy_k}{dt} = -\sum_{i,j} \Gamma_{ij}^k y_i y_j \end{cases}, \quad \text{for } k = 1, \ldots, n, \tag{3.2}$$

on the coordinate system $(x_1, \ldots, x_n, y_1, \ldots, y_n)$ on TU. Hence, solving the second order system of equations (3.1) on U is equivalent to solving the first order system (3.2) on TU.

Lemma 8. *There is an unique vector field G on TM whose flow linear are precisely the solutions to the system of equations* (3.2).

Proof. If such a vector field G exists, it must be unique. Indeed, let $(U, \mathbf{x})$ be a coordinate system on M. Then, on these local coordinates, G is uniquely defined be (3.2).

To prove existence of the field G, define it on each coordinate system by the local expression given by (3.2). Since in the intersection of two coordinate charts the local expressions of G agree, the vector field G is well-defined on all TM. □

Another way to state Lemma 8 above is to say that there exists an unique vector field G on TM whose flow lines are of the form $t \mapsto (\gamma(t), \gamma'(t))$, where γ is a geodesic on M.

Definition 16. *The vector field G defined on Lemma 8 is called* ***geodesic field on*** TM *and its solution is called the* ***geodesic flow on*** TM.

Applying the Fundamental Theorem of ODE's (see Example 1 for a brief discussion) to the geodesic field G on TM at the point $(p, 0) \in TM$, we obtain the following:

Theorem. *For each $p \in M$ there is an open set $\mathcal{U}$ of TU, where (U, x) a coordinate system around p, and such that $(p, 0) \in \mathcal{U}$, and there are a positive number $\delta > 0$ and a C^∞−map $g\colon (-\delta, \delta) \times \mathcal{U} \to TU$ such that $t \mapsto g_t(q, w)$ is the unique trajectory of G that satisfies the initial condition $g_0(q, w) = (q, w)$, for each $(q, w) \in \mathcal{U}$.*

The Geometry of TM

We make a small detour into symplectic geometric and Hamiltonian dynamics, in order to obtain new information on the geodesic flow. More precisely, we show that the geodesic flow is Hamiltonian and, hence, has no divergence. This will be used to show that the geodesic flow cannot be an example of suspension flow.

Definition 17 (Symplectic manifold)**.** *A* ***symplectic form*** *on a manifold M is a nondegenerate, closed 2-form ω on M.*

Proposition 12. *Let M be a manifold and $\omega \in \Omega^2(M)$ be a 2-form on M. Then, ω is nondegenerate if and only if M is even-dimensional, say $\dim M = 2n$, and $\omega^n = \omega \wedge \cdots \wedge \omega$ is a volume form on M.*

Proof. For a proof, see Proposition 3.1.5, p.166, and Proposition 3.1.3, p. 165, of [AM08]. □

As a consequence we observe that, if $\omega \in \Omega^2(M)$ is nondegenerate, M is orientable. Denote the standard volume by

$$\Omega_\omega := \frac{(-1)^{n/2}}{n!}\omega^n.$$

The first example of symplectic manifold is the euclidean space $\mathbb{R}^{2n}$ with the following form: if $(x_1, \ldots, x_n, y_1, \ldots, y_n)$ are the coordinates of $\mathbb{R}^{2n}$ define

$$\omega = \sum_{i=1}^{n} dx_i \wedge dy_i.$$

A less obvious symplectic manifold is the tangent bundle TM of a Riemannian manifold $(M, \langle\cdot,\cdot\rangle)$. In order to define a form on TM we need to study first its tangent bundle, i.e., TTM.

Let $\pi\colon TM \to M$ be the canonical projection, i.e., if $\theta = (p, v) \in T_pM \subset TM$ then $\pi(\theta) = p$. For each $\theta = (p, v) \in TM$, the tangent space $T_\theta TM$ has a decomposition on subspaces that will be very useful for us.

Define the vertical subspace $V(\theta)$ as the subspace of $T_\theta TM$ of vectors that, for $t = 0$, are tangent to curves $X\colon (-\varepsilon, \varepsilon) \to TM$ of the form

$$t \mapsto (p, v + tw),$$

where $w \in T_pM$. In other words, $V(\theta) := \ker d\pi_\theta$.

Now, for the same $\theta = (p, v) \in TM$, we define another subspace of $T_\theta TM$ as the kernel of a map $K_\theta\colon T_\theta TM \to TM$ as follows: let $\xi \in T_\theta TM$ and $X(t)$ be a curve on TM with $X(0) = \theta$ and $X'(0) = \xi$. Call $\gamma(t) = \pi \circ X(t)$ and define:

$$K_\theta(\xi) = \nabla_{\gamma'} X(0),$$

where ∇ is the Riemannian connection of $(M, \langle\cdot,\cdot\rangle)$.

This map K_θ is well-defined (i.e., it does not depend on the curve X) and is linear (see [Kni02], p.455, for details). Set $H(\theta) = \ker K_\theta$.

Lemma 9. *For each $\theta = (p, v) \in TM$, the following hold:*

(i) $H(\theta) \cap V(\theta) = \{0\}$*;*

(ii) $\dim H(\theta) = \dim V(\theta) = \dim M$*;*

(iii) $T_\theta TM = H(\theta) \oplus V(\theta)$*;*

(iv) *The restrictions* $d\pi_\theta\big|_{H(\theta)}\colon H(\theta) \to T_pM$ *and* $K_\theta\big|_{V(\theta)}\colon V(\theta) \to T_pM$ *are linear isomorphisms.*

Proof. Observe that item (iii) is a direct consequence of items (i) and (ii). For proofs of items (i), (ii), and (iv), see [Kni02], p.455. □

As a direct consequence of the above lemma, the map $(d\pi_\theta, K_\theta)\colon T_\theta TM \to T_pM \times T_pM$ defined by $\xi \mapsto (d\pi_\theta(\xi), K_\theta(\xi))$ is a linear isomorphism between $T_\theta TM$ and $T_pM \times T_pM$. We the identify

$$H(\theta) \simeq \{(w, 0) \mid w \in T_pM\},$$

and

$$V(\theta) \simeq \{(0, w) \mid w \in T_pM\},$$

and call $H(\theta)$ the horizontal subspace of $T_\theta TM$ and $V(\theta)$ the vertical subspace of $T_\theta TM$. With the above identification we write $\xi \in T_\theta TM$ as $\xi = (\xi_h, \xi_v)$, where $\xi_h = d\pi_\theta(\xi)$ and $\xi_v = K_\theta(\xi)$.

An useful application of the identification between $T_\theta TM$ and $T_pM \times T_pM$ is that, in this notation, the geodesic vector field G (see Definition 16) can be written as

$$G(\theta) = (v, 0), \tag{3.3}$$

for each $\theta = (p, v) \in TM$ (for details see p.455 of [Kni02]).

With the decomposition $T_\theta TM = H(\theta) \oplus V(\theta)$ in hand, we can define a Riemannian metric on TM that makes the subspaces $H(\theta)$ and $V(\theta)$ orthogonal:

Definition 18 (Sasaki metric)**.** *At each $\theta = (p, v) \in TM$, define the Sasaki metric at θ by*

$$\begin{aligned}\langle \xi, \eta \rangle_\theta^{Sas} &= \langle d\pi_\theta(\xi), d\pi_\theta(\eta) \rangle_{\pi(\theta)} + \langle K_\theta(\xi), K_\theta(\eta) \rangle_{\pi(\theta)} \\ &= \langle \xi_h, \eta_h \rangle_{\pi(\theta)} + \langle \xi_v, \eta_v \rangle_{\pi(\theta)} \\ &= \langle \xi_h, \eta_h \rangle_p + \langle \xi_v, \eta_v \rangle_p,\end{aligned}$$

for all $\xi, \eta \in T_\theta TM$.

Also from the decomposition $T_\theta TM = H(\theta) \oplus V(\theta)$ we can use the Sasaki metric to define a symplectic form ω on TM. For $\theta = (p, v) \in TM$, define ω_θ as

$$\omega_\theta(\xi, \eta) = \langle K_\theta(\xi), d\pi_\theta(\eta) \rangle_{\pi(\theta)} - \langle K_\theta(\eta), d\pi_\theta(\xi) \rangle_{\pi(\theta)} = \langle \xi_v, \eta_h \rangle_p - \langle \eta_v, \xi_h \rangle_p. \tag{3.4}$$

We are going to show that ω is a symplectic form on TM. It is clearly nondegenerate. To see it is closed, define a $1-$form Θ on TTM by taking, for each $\theta = (p, v) \in TM$, the map $\Theta \colon T_\theta TM \to \mathbb{R}$ defined by $\Theta_\theta(\xi) = \langle \theta, d\pi_\theta(\xi) \rangle$. Writing $\xi = (\xi_h, \xi_v) \in H(\theta) \times V(\theta)$, we have: $\Theta_\theta(\xi) = \Theta_\theta(\xi_h, \xi_v) = \langle v, \xi_h \rangle_p$.

Lemma 10. *The $2-$form ω and the $1-$form Θ satisfy:*

$$d\Theta = \omega.$$

In particular, ω is closed.

Proof. See Lemma 1.2, p.456, of [Kni02]. □

Example 11. *From Lemma 10 above we conclude that the pair (TM, ω), where ω is the differential form defined by* (3.4)*, is a symplectic manifold.*

The form ω defined by 3.4 has a very important property for our goals.

Hamiltonian Flows

In this subsection we show that the geodesic flow on TM is a particular case of a more general type of flows, namely the Hamiltonian flows.

Proposition 13. *For each $\theta = (p, v) \in TM$, the form ω defined by* (3.4) *is invariant under the geodesic flow $g_t \colon TM \to TM$, i.e., for all $t \in \mathbb{R}$ and $\xi, \eta \in T_\theta TM$, we have:*

$$\omega_\theta((dg_t)_\theta(\xi), (dg_t)_\theta(\eta)) = \omega_\theta(\xi, \eta).$$

Proof. See Lemma 1.3, p.457, of [Kni02]. □

Definition 19. *Let (M, ω) be a symplectic manifold and $H \colon M \to \mathbb{R}$ be a given C^r ($r \geq 1$) function. The Hamiltonian vector field X_H of the function H is determined by*

$$\omega(X_H, Y) = dH \cdot Y,$$

for all vector field $Y \in \mathfrak{X}^{r-1}(M)$. If we write $\imath$ for the contraction operation, we may write:

$$\imath_{X_H}(\omega) = dH.$$

Proposition 14. *Let $(q^1, \dots, q^n, p_1, \dots, p_n)$ be the canonical coordinates for ω (so that $\omega = \sum_{i=1}^n dq^i \wedge dp_i$). Then, in these coordinates,*

$$X_H = \left(\frac{\partial H}{\partial p_i}, -\frac{\partial H}{\partial q^i} \right) = J \cdot dH,$$

where $J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$. Thus, $(q(t), p(t))$ is an integral curve of X_H if and only if the following equations hold:

$$\begin{cases} \frac{dq^i}{dt} = \frac{\partial H}{\partial p_i} \\ \frac{dp_i}{dt} = -\frac{\partial H}{\partial q^i} \end{cases}, \tag{3.5}$$

for $i = 1, \dots, n$.

Proof. See Proposition 3.3.2 at p.187 of [AM08] for a proof. □

The equations 3.5 are called ***Hamilton equations*** of the system (M, ω, X_H).

Now we present a proposition that is the central property that we want the geodesic flow to satisfy:

Proposition 15. *Let (M, ω, X_H) be a Hamiltonian system and φ_t be the flow of X_H. Then,* $\operatorname{div} X_H = 0$ *and, for each t, $\varphi_t^* \omega = \omega$ and φ_t preserves the volume Ω_ω.*

Proof. For a proof of this proposition see Proposition 3.3.4, p. 188 of [AM08], together with the proof of Liouville's Theorem, at p. 69 of [Arn13]. □

Definition 20 (Volume measure)**.** *Define on M the **volume measure** m **induced by the volume form**, i.e., for measurable set B contained in some coordinate domain we set:*

$$m(B) = \int_B \Omega_\omega.$$

Theorem 12 (Liouville)**.** *Let $\varphi_t \colon M \to M$ be the flow associated to a C^1 vector field X on M. Then, φ_t preserves the volume of M if and only if* $\operatorname{div} X = 0$.

Proof. For a proof see Theorem 1.3.7, p.21, of [VO16] or, alternatively, Liouville's Theorem, at p. 69 of [Arn13]. □

Corollary 5. *Every Hamiltonian flow preserves the volume measure.*

Example 12. *Let $(M, \langle \cdot, \cdot \rangle)$ be a symplectic manifold and let $H \colon TM \to \mathbb{R}$ be a map defined by*

$$H(p, v) = \frac{1}{2} \langle v, v \rangle_p.$$

Then, a curve $\gamma \colon I \to M$ is an integral curve of X_H if, and only if, γ is a geodesic. In particular, this shows that the geodesic vector field is Hamiltonian.

Indeed, let $\theta = (p, v) \in TM$ and $\xi \in T_\theta TM$. If we write $X_H = (X, Y)$ and $\xi = (\xi_h, \xi_v)$ we have:

$$\omega_\theta(\xi, X_H) = \langle \xi_v, X \rangle_p - \langle \xi_h, Y \rangle_p.$$

Now, let $Z\colon (-\varepsilon, \varepsilon) \to TM$ be a curve with $Z(0) = \theta$ and $Z'(0) = \xi$. By writing $\gamma(t) = \pi \circ Z(t)$ we may also write $Z'(0) = (\gamma'(0), \nabla_\gamma Z(0))$. So,

$$\begin{aligned} dH_\theta(\xi) &= \left.\frac{d}{dt}\right|_{t=0} H(Z(t)) \\ &= \frac{1}{2}\left.\frac{d}{dt}\right|_{t=0} \langle Z(t), Z(t)\rangle \\ &= \langle Z(0), \nabla_\gamma Z(0)\rangle \\ &= \langle v, \xi_v\rangle \\ &= \langle v, \xi_v\rangle - \langle \xi_h, 0\rangle \\ &= \omega_\theta(G(\theta), \xi), \end{aligned}$$

by the representation of the geodesic field G given in (3.3).

Since we are interested on dynamical properties of the geodesic flow $g_t\colon TM \to TM$, it is worth to study it on a compact manifold. Since, even if M is, TM need not to be compact, we should study the flow somewhere else.

Observe that, as we showed in the beginning of Section 3.2, the norm of γ' is constant. Therefore, if we want to analyze the geodesic flow, we can restrict ourselves to unit vectors. To do so, define T^1M, the unit tangent bundle of M, by:

$$T^1M = \{(p, v) \mid x \in M,\ v \in T_pM \text{ and } \|v\| = 1\}.$$

Now, T^1M is a compact $2n-1$ manifold and the restriction of g_t to T^1M is well defined in the sense that $g_t(T^1M) \subseteq T^1M$.

Next, we present a classical theorem due to Anosov (see [Ano67]) that describes the dynamics of the geodesic flow on the unit tangent bundle of a negatively curved complete manifold.

Theorem 13. *Let $(M, \langle\cdot,\cdot\rangle)$ be a complete Riemannian manifold such that there are constants $\beta \geq \alpha > 0$ satisfying $-\beta^2 \leq K \leq -\alpha^2$ for all its sectional curvatures. Then its geodesic flow is an Anosov flow.*

Proof. See Proposition 3.2, p.474, of [Kni02]. □

Corollary 6. *The geodesic flow $g_t\colon T^1M \to T^1M$ on the unit tangent bundle of a compact Riemannian manifold M with constant negative curvature is an Anosov flow.*

It may be interesting to say some words on what happens outside the curvature hypothesis, i.e., the hypothesis that all sectional curvatures are bounded as follows:

$$-\beta^2 \leq K \leq -\alpha^2,$$

where $\beta \geq \alpha > 0$ are constant.

Outside these hypothesis there are all sort of behavior: from a result due to Klingenberg from the 70's (see [Kli74]), neither the sphere nor the torus admit Riemannian structures whose geodesic flows are Anosov. This means that for positive curvature (such as in the case of the sphere), we could never formulate a general result such as Theorem 13.

If, instead of positive curvature, we allow non-negative curvature, one can find examples! Donnay and Pugh present in [DP03] examples of embedded compact surfaces in $\mathbb{R}^3$ such that its geodesic flow is Anosov. So in non-negative curvature we have both Anosov and non-Anosov behaviors.

In fact, the proper setting of the curvature hypothesis is indeed the one presented in Theorem 13. Even in strictly negative curvature, there are counter-examples. In [MR20], Ítalo Melo and Sergio Romaña show that there are a class of embedded surfaces in $\mathbb{R}^3$ of negative curvature, such that its geodesic flow is not Anosov. The issue here is that either their curvature is unbounded from below or is asymptotically zero. In particular, the hypotheses on Theorem 13 are the correct ones, in the sense that we could find counter-examples otherwise.

Example 13. *Let S be a compact orientable surface with genus at least 2. It follows from Proposition 4.5, p.167 of [DF92], that we can endow it with a Riemannian metric with constant negative curvature. This gives explicit examples of Anosov geodesic flows.*

Since the suspension of an Anosov diffeomorphism is already an example of Anosov flow, one could ask if the new and the old examples are, indeed, different.

Proposition 16. *A geodesic flow on the unit tangent bundle T^1M of a n-dimension manifold M is never a suspension.*

The proof of this fact follows from the following lemma:

Lemma 11. *On T^1M, there is no codimension one smooth closed submanifold that is transverse to the geodesic flow g_t.*

The proof of Lemma 11 we present here follows what is done in Lemma 2.52, in p.49 of [Pat12]. Before we start, we need to clarify one point: since $\dim T^1M = 2n-1$, T^1M has odd dimension and hence, it cannot admit a symplectic form (and of course, cannot be a symplectic manifold). Instead, it admits a *contact form*.

Definition 21 (Contact manifold)**.** *A $1-$form α on an orientable manifold N of dimension $2n-1$ is called a **contact form** if the $(2n-1)-$form*

$$\alpha \wedge (d\alpha)^{n-1}$$

*never vanishes. A pair (N, α) of an odd-dimensional manifold and a contact form is called a **contact manifold**.*

One can define a contact one form α on T^1M as follows: for $\theta = (p, v) \in T^1M$ and $\xi \in T_\theta T^1M$ we set

$$\alpha_\theta(\xi) = \langle \xi, G(\theta)\rangle_\theta^{Sas} = \langle d\pi_\theta(\xi), v\rangle_p, \tag{3.6}$$

where G is the geodesic vector field and $\pi\colon TM \to M$ is the canonical projection.

Proposition 17. *The $1-$form α and the symplectic form ω on TM are related by the following formula:*

$$\omega = -d\alpha. \tag{3.7}$$

Proof. For a proof, see Proposition 1.24 in [Pat12]. □

Corollary 7. *The $1-$form α is a contact form on T^1M.*

Proof. See Corollary 1.29 in [Pat12]. □

Given a contact manifold (N, α) there is an unique vector field X on N, that satisfies:

$$\begin{cases} \imath_X \alpha = 1 \\ \imath_X d\alpha = 0 \end{cases}.$$

This vector field X called ***characteristic vector field*** (or ***Reeb vector field***) and its flow is called ***characteristic vector flow***.

Note that the characteristic flow preserves α. Indeed, by Cartan's magic formula, we know that $\mathcal{L}_X\alpha = d\imath_X\alpha + \imath_X d\alpha = 0$ where $\mathcal{L}_X\alpha$ stands for the Lie derivative. In particular,

$$0 = \mathcal{L}_X\alpha = \left.\frac{d}{dt}\right|_{t=0} (\varphi_t^*\alpha),$$

i.e., $\varphi_t^*\alpha = \alpha$.

In the case $N = T^1M$ and α is the contact form defined in (3.6), we have that the characteristic flow of α is the geodesic flow. Indeed, for $\theta = (p, v) \in T^1M$, we have

$$\alpha_\theta(G(\theta)) = \langle d\pi_\theta(G(\theta)), v\rangle = \langle v, v\rangle = 1,$$

i.e., $\imath_G\alpha_\theta = 1$. Also, for $\xi \in T_\theta T^1M$, we have:

$$\imath_G(d\alpha_\theta)(\xi) = d\alpha_\theta(G(\theta), \xi) = -\omega_\theta(G(\theta), \xi) = -dH_\theta(\xi) = 0,$$

since $dH_\theta(\xi) = \langle K_\theta(\xi), v\rangle$ and since $\xi \in T_\theta TM$ lies in $T_\theta T^1M$ if and only if $K_\theta(\xi) = 0$. From the previous paragraph, the geodesic flow preserves the contact form α on T^1M.

Definition 22 (Liouville measure)**.** *The volume measure induced by the volume form $\alpha \wedge (d\alpha)^{n-1}$ on T^1M is called the **Liouville measure**. Whenever M has finite volume, the volume form $\alpha \wedge (d\alpha)^{n-1}$ has finite integral over T^1M and, hence, gives rise to a probability measure.*

After introducing the contact form α on T^1M, we are ready to prove Lemma 11:

Proof of Lemma 11. Suppose Σ is a codimension one closed submanifold in T^1M and that Σ is transverse to the geodesic flow g_t, that is, at each point $\theta = (p, v) \in \Sigma \subseteq T^1M$, we have:

$$T_\theta T^1M = T_\theta\Sigma \oplus \langle G(\theta)\rangle.$$

Consider now the symplectic form ω on TM defined on (3.4). Let $i\colon \Sigma \to T^1M$ be the inclusion map and let $i^*\omega$ denote the pull-back of the form ω by the inclusion i. We claim the pair $(\Sigma, i^*\omega)$ is a symplectic manifold.

In order to prove the claim, recall that the geodesic vector field G is the characteristic vector field of the contact form α, defined in (3.6). Since Σ is transverse to the geodesic flow, we have $T\Sigma \subseteq \ker\alpha$. In fact, since α is a non-degenerate contact form:

$$\dim\ker\alpha = (2n-1) - 1 = \dim T\Sigma,$$

and hence $T\Sigma = \ker \alpha$.

On the other hand, for a contact form α, the restriction of $d\alpha$ to $\ker \alpha$ is non-degenerate. Then, $d\alpha$ restricted to Σ is symplectic. Since we know, from Proposition 17, that $-d\alpha = \omega$, the pair $(\Sigma, i^*\omega)$ is symplectic.

A consequence of this conclusion is that, since Σ is a codimension one manifold in T^1M, and $i^*\omega^{n-1}$ is nondegenerate $2n-2$ form on Σ, $i^*\omega^{n-1}$ a volume form in Σ. So,

$$\int_\Sigma i^*\omega^{n-1} \neq 0.$$

On the other hand, by definition of ω, ω^{n-1} coincides (up to a sign) with $(d\alpha)^{n-1} = d(\alpha \wedge (d\alpha)^{n-2})$. Then, since we are supposing Σ to be without boundary, Stokes' theorem implies:

$$\begin{aligned}\int_\Sigma i^*\omega^{n-1} &= \int_\Sigma i^*\left(d\left(\alpha \wedge (d\alpha)^{n-2}\right)\right)\\ &= \int_\Sigma d\left(i^*\left(\alpha \wedge (d\alpha)^{n-2}\right)\right)\\ &= \int_{\partial\Sigma} i^*(\alpha \wedge (d\alpha)^{n-2}) = 0,\end{aligned}$$

i.e., $\int_\Sigma i^*\omega^{n-1} = 0$. This is a contradiction. □

Proof of Proposition 16. Suppose, by contradiction, that the geodesic flow $g_t \colon T^1M \to T^1M$ is a suspension flow, i.e., that there exists a diffeomorphism $f \colon \Sigma \to \Sigma$ from a $n-1$-dimensional closed manifold to itself such that:

(i) $T^1M = \Sigma \times \mathbb{R}/\sim$, where $\sim$ is the equivalence relation defined in the construction of the suspension flow (in Subsection 3.1);

(ii) the geodesic vector field $X = \frac{\partial g_t}{\partial t}$ is transverse to Σ;

(iii) the Poincaré map of g_t is conjugated to f.

Since $\Sigma \simeq \Sigma \times \{0\}$, we can think of Σ as a codimension one closed submanifold in T^1M. By property (ii) above, the vector field X is transverse to Σ, providing a contradiction with Lemma 11. Hence, the geodesic flow on T^1M can never be a suspension of a diffeomorphism.

□

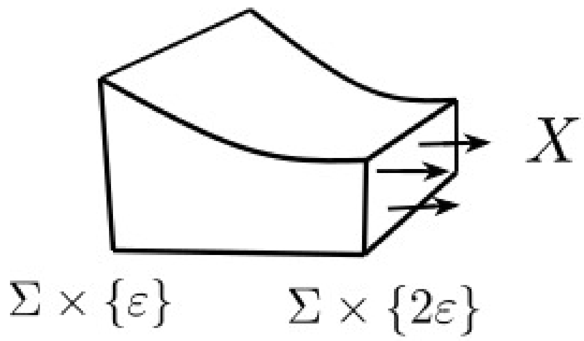

Figure 3.6: A small portion of the manifold $\widetilde{\Sigma}$, for small $\varepsilon > 0$.

As we claimed at the very end of Subsection 3.1, we now prove that an Anosov geodesic flow is an example of a transitive flow.

Theorem 14. *An Anosov geodesic flow $g_t \colon T^1M \to T^1M$ is always transitive.*

Proof. From Theorem 11, proving g_t is transitive is equivalent as showing that $\Omega(\varphi) = M$. Since the geodesic flow is a Hamiltonian flow, Corollary 5 implies that the Liouville measure μ on T^1M is invariant by g_t, i.e., $(g_t)_*\mu = \mu$.

From Poincaré Recurrence Theorem, $\mu-$almost every point in M is recurrent for g_t.[3] Since, the Liouville measure μ has full support, i.e., $\operatorname{supp}(\mu) = M$, we have:

$$M = \operatorname{supp}(\mu) = \overline{\operatorname{Rec}(g_t)} \subseteq \Omega(g_t) \subseteq M,$$

and then $\Omega(g_t) = M$, proving the claim. □

At this point, we presented two examples of Anosov flows: the geodesic flow on negatively curved manifolds and the suspension flow of an Anosov diffeomorphism. Proposition 16 shows that they are, indeed, two different examples. However, one can ask whether they are dynamically different: can they be topologically conjugated?

The next chapter is devoted to prove a theorem that show this question has a negative answer. Placing together Theorem 14 with two results from Chapter 4 (namely Theorem 15 and Proposition 19), we prove that an Anosov geodesic flow is topologically mixing. This is never the case of a suspension flow with constant height function, as we will discuss in further detail through Chapter 4.

The case of surfaces and the horocycle flow

Whenever the manifold M is a $2-$dimensional surface S, the geodesic flow acts on the $3-$dimensional manifold T^1S. If we also ask for S to have constant negative curvature, Corollary 6 implies that the geodesic flow $g_t \colon T^1S \to T^1S$ is an Anosov flow.

Since it is an Anosov flow, the tangent bundle of T^1S admits a decomposition of the form

$$T(T^1S) = E^u \oplus E^c \oplus E^s,$$

on unstable, center, and stable spaces. The Stable Manifold Theorem (Theorem 5) guarantees that, through each point $(x, v) \in T^1S$, we have well-defined $1-$dimensional integral manifolds to E^u and E^s: the unstable and stable manifolds $W^u(x, v)$ and $W^s(x, v)$.

These can be parametrized as flows that we call **stable and unstable horocycle flows**, respectively.

By calling the stable horocycle flow $h_s \colon T^1S \to T^1S$, we have the interesting property that relates it with the geodesic flow g_t:

Proposition 18. *Let $\theta = (p, v) \in T^1S$ and $s, t \in \mathbb{R}$. Then,*

$$g_t \circ h_s \circ g_{-t}(\theta) = h_{se^{-t}}(\theta).$$

Proof. For a proof see, for example, Property 3.3, p. 119, in [Dal10]. □

[3] See, for example, Theorem 1.2.4 on [VO16].

CHAPTER 4

Minimality of invariant manifolds

In the spirit of the examples given in Chapter 3, we keep on studying transitive Anosov flows. There, we introduced two different examples of Anosov flows: the geodesic flow on negatively curved manifolds and the suspension flow of an Anosov diffeomorphism. From Proposition 16 we concluded that they are, indeed, two different examples. In this chapter we prove even more: their dynamics are also of different nature.

The main theorem of this chapter establishes a dichotomy for transitive Anosov flows. It shows that such a flow is either a suspension flow, or that it is minimal, i.e., that the leafs of the stable and unstable foliations are dense. Next we show that this minimality property implies the topologically mixing property. As a consequence, we conclude that a geodesic Anosov flow is topologically mixing and, therefore, cannot be topologically conjugated to a suspension flow.

Throughout this Chapter 4, M will be a closed and connected Riemannian manifold and $\varphi_t \colon M \to M$ an Anosov flow of class C^r (with $r \geq 1$) on M. We remember that, by definition of Anosov flow, there exists a φ_t-invariant decomposition

$$TM = E^u \oplus E^c \oplus E^s,$$

i.e., each subspace E^u, E^c and E^s, is preserved by $d\varphi_t$, E^c is the space generated by the vector field associated to the flow, and the vectors on E^u are exponentially expanded by $d\varphi_t$ and the vectors on E^s are exponentially contracted by $d\varphi_t$. Remember also that the spaces E^u and E^s are called unstable and stable spaces, respectively.

By the Stable Manifold Theorem (see Theorem 5 above), we know that each of these tangent bundles are uniquely integrable and give rise to foliations tangent to it. For each bundle we have the analogue foliation as follows:

Bundle	Foliation
$E^u \oplus E^c$	$\mathcal{F}^u$
$E^s \oplus E^c$	$\mathcal{F}^s$
E^u	$\mathcal{F}^{uu}$
E^s	$\mathcal{F}^{ss}$

Now, since the bundles are originated by the dynamics of the Anosov flow φ_t, it would be a natural work to look out for this new interplay between that flow and the foliation we have obtained by integrating those tangent bundles. On that direction, J. Plante shows on his 1972 paper [Pla72] an interesting property that each leaf of these foliations must satisfy:

Theorem 15. *Let $\varphi_t \colon M \to M$ be an Anosov flow on a compact and connected Riemannian manifold. If φ_t is transitive then all (weak) stable and unstable manifolds are dense on M and, if **some** strong stable or unstable manifold is **not** dense, then φ_t is the suspension of some diffeomorphism defined on a codimension one compact submanifold.*

Note that Theorem 15 reveals a striking difference between transitive Anosov flows and transitive Anosov diffeomorphisms. Indeed, as we show in Appendix A, for a transitive Anosov diffeomorphism $f \colon M \to M$, every stable and unstable manifolds are dense on M (i.e., f is minimal) and f is topologically mixing.

Warning: during the text we will call a flow **minimal** in two different contexts. It will have the meaning that all its orbits are dense, that is the notion for a general dynamical system. Also, it will have the meaning that all stable and unstable leafs are dense, that will be used for Anosov flows. Observe that an Anosov flow cannot be minimal in the sense that all its orbits are dense, since it has periodic points. From the context, it will be clear to which meaning we are referring to.

As Theorem 15 indicates, the case for flows may be different. In Example 10 we have seen that indeed there are examples of transitive Anosov flows that are the suspension flow of some diffeomorphism. Whenever that is the case, the flow cannot be mixing. More precisely we have the following propositions:

Proposition 19. *Let $\varphi_t \colon M \to M$ be a transitive Anosov flow on a closed and connected Riemannian manifold M. If φ_t is minimal, then φ_t is topologically mixing.*

Proof. The proof of this proposition is very similar to what is done in the case of diffeomorphisms. In fact, we simply adapt the proof given in Appendix A for Theorem 26.

Lemma 12. *If every (strong) unstable manifold is dense in M, then for every $\varepsilon > 0$ there is $R = R(\varepsilon) > 0$ such that every ball of radius R in every (strong) unstable manifold is dense on M.*

Proof. Let $x \in M$ and notice that $W^{uu}(x) = \bigcup_{R>0} W^{uu}_R(x)$, where $W^{uu}_R(x)$ represents the local (strong) unstable manifold of diameter R around x. Since $W^{uu}(x)$ is dense, there is $R(x) > 0$ such that $W^{uu}_{R(x)}(x)$ is $\varepsilon/2-$dense on M. Moreover, since the foliation W^{uu} is continuous, there exists a $\delta(x) > 0$ such that $W^{uu}_{R(x)}(y)$ is $\varepsilon-$dense for all $y \in B_{\delta(x)}(x)$.

Since we are supposing M compact, there is a finite subcollection $\mathcal{B}'$ of the collection $\mathcal{B} := \{B_{\delta(x)}(x) \mid x \in M\}$ that still covers M. By taking R to be the maximum $R(x)$ associated with the balls on $\mathcal{B}'$, we obtain an uniform radius R such that every R-ball in some unstable manifold is dense on M. $\square$

Now let $U, V \subseteq M$ be non-empty open sets and $x \in U$. Inside U, consider $D \subseteq W^{uu}(x) \cap U$ a small disc of (strong) unstable manifold; and inside V consider a small ball B_ε of radius ε.

Since D lies inside the (strong) unstable foliation there is $\lambda > 1$ such that $d(\varphi_t(x), \varphi_t(y)) > \lambda^t d(x, y)$ for all $t \in \mathbb{R}$ and all $x, y \in D$. So, there exists $T > 0$ such that $\operatorname{diam}(\varphi_T(D)) > 2R$. Hence, by Lemma 12 above, $\varphi_T(D)$ is $\varepsilon-$dense on M. Therefore, $\varphi_T(D) \cap B_\varepsilon \neq \emptyset$ and, in particular, $\varphi_T(U) \cap V \neq \emptyset$.

Moreover, since $\varphi_T(D) \subseteq \varphi_t(D) \subseteq W^{uu}(x)$ for all $t > T$,

$$\operatorname{diam}(\varphi_t(D)) > \operatorname{diam}(\varphi_T(D)) > 2R,$$

for all $t > T$. And then, $\varphi_t(U) \cap V \neq \emptyset$ for all $t \geq T$, proving that φ is topologically mixing.

□

On the other hand, as we have proved in Proposition 8, if we know that φ_t is the suspension flow of an Anosov diffeomorphism, then it cannot be topologically mixing.

Throughout the rest of this Chapter 4 we present the proof of Theorem 15, that is given in [Pla72].

4.1 Density of (weak) invariant manifolds

In this section we prove the first part of our claim:

Theorem 16. *Let M be a compact and connected Riemannian manifold and $\varphi_t\colon M \to M$ be an Anosov flow of class C^r (with $r \geq 1$) on M. If $\Omega(\varphi_t) = M$, then:*

$$\overline{W^s(x)} = \overline{W^u(x)} = M,$$

for all $x \in M$.

Proof. We prove the theorem for $\overline{W^u(x)}$ (the stable case is similar). Since we are assuming the manifold M to be connected, we restrict ourselves to prove that the closed non-empty set $\overline{W^u(x)}$ is also open.

In order to do that, let $z \in \overline{W^u(x)}$ and let $N(z) \subseteq M$ be a product neighborhood for z in M. We shall prove that

$$\operatorname{Per}(\varphi) \cap N(z) \subseteq \overline{W^u(x)}$$

and to do so, consider $p \in \operatorname{Per}(\varphi) \cap N(z)$ (which exists by Corollary 4). Moreover, let $q \in N(z)$ be the intersection point of $W^s_{loc}(p)$ and $W^{uu}_{loc}(z)$, i.e., $\{q\} = W^s_{loc}(p) \cap W^{uu}_{loc}(z)$.

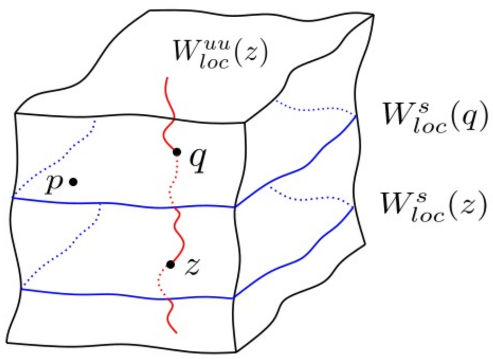

Figure 4.1: The product neighborhood $N(z)$.

Since $p \in \operatorname{Per}(\varphi)$ and $q \in W^s_{loc}(p)$, the $\omega-$limit set of q coincides with the orbit of p:

$$\omega(q) = \mathcal{O}(p),$$

and hence $p \in \omega(q)$. On the other hand, since $q \in W^{uu}_{loc}(z)$ we obtain $q \in W^u(z) \subseteq \overline{W^u(x)}$ and, because $\overline{W^u(x)}$ is saturated by the flow φ_t, we must have $\omega(q) \subseteq \overline{W^u(x)}$. Now, since $p \in \omega(q)$, we conclude that $p \in \overline{W^u(x)}$.

Hence, we have shown that for every point $z \in \overline{W^u(x)}$ there is an open neighborhood $N(z)$ such that

$$N(z) \subset \overline{N(z)} = \overline{\mathrm{Per}\,(\varphi) \cap N(z)} \subseteq \overline{W^u(x)},$$

proving that $\overline{W^u(x)}$ is, more than a non-empty closed subset of M, also an open one. This shows that $\overline{W^u(x)} = M$. □

4.2 Minimality of the strong unstable foliation

The minimality of the strong stable and unstable manifolds is a little more subtle: it may not even occur. However, if this happens in our setting, we have a side effect that may be useful in several occasions[1]:

Theorem 17. *Let M be a compact and connected Riemannian manifold and $\varphi_t \colon M \to M$ be an Anosov flow of class C^r (with $r \geq 1$) on M. If $\Omega(\varphi_t) = M$, then there are **exactly** two possibilities:*

(i) *The strong unstable (stable) manifolds through **every** point $x \in M$ is dense on M, i.e.,*

$$\overline{W^{uu}(x)} = M,$$

for every $x \in M$;

(ii) *If there exists $x \in M$ such that $\overline{W^{uu}(x)} \neq M$, then there exists an Anosov diffeomorphism $f \colon K \to K$ on a compact codimension one C^1-submanifold $K \subset M$ such that the Anosov flow φ_t is a suspension of this diffeomorphism.*

The proof is a little bit long and follows through several steps, which we state without proof for a moment in order to first prove Theorem 17. The theorem states a dichotomy. So let us first suppose that there exist such a point $x \in M$ such that its strong unstable manifold is not dense, i.e., such that $\overline{W^{uu}(x)} \neq M$.

(a) First of all we can suppose that x is, in fact, a periodic point of φ_t, since

Lemma 13. *If $W^{uu}(p)$ is dense in M for all periodic point p then $W^{uu}(x)$ is dense in M for all $x \in M$.*

Since we are supposing that there exists a point $x \in M$ such that $\overline{W^{uu}(x)} \neq M$, this Lemma implies the existence of a periodic point $p \in M$ such that $\overline{W^{uu}(p)} \neq M$, i.e., there exists a point $p \in \mathrm{Per}\,(\varphi)$ such that its strong unstable manifold is not dense on M.

(b) In this case, where there exists $p \in \mathrm{Per}\,(\varphi)$ such that $\overline{W^{uu}(p)} \neq M$, we obtain a fibration:

Lemma 14. *If $p \in \mathrm{Per}\,(\varphi)$ is such that $\overline{W^{uu}(p)} \neq M$, then:*

- *M is a fiber bundle;*

[1]See, for instance, Section 4.4.

- *with base space S^1;*
- *with fiber $K = \overline{W^{uu}(p)}$;*
- *and φ_t is the suspension flow of an homeomorphism $f\colon K \to K$.*

It is important to notice that this lemma does not resume the proof of Theorem 17: at this point, this fibration only exists at a topological level and we still want two major improvements. First we must check that the set $K = \overline{W^{uu}(p)}$ is, in fact, a compact C^1-submanifold of codimension one in M; second, we need the homeomorphism claimed in Lemma 14 to be an Anosov diffeomorphism. In order to complete the proof we need two more lemmas:

(c) The first one translate the condition $\overline{W^{uu}(p)} \neq M$ that has to do with a single leaf with a global property on the whole foliations $\mathcal{F}^{uu}$ and $\mathcal{F}^{ss}$:

Lemma 15. *If some strong unstable (stable) manifold is not dense on M, then the foliations $\mathcal{F}^{uu}$ and $\mathcal{F}^{ss}$ are jointly integrable.*

(d) The second and last lemma allow us to obtain an integrability condition on the bundle we are studying:

Lemma 16. *The foliations $\mathcal{F}^{uu}$ and $\mathcal{F}^{ss}$ are jointly integrable if and only if the splitting $E^u \oplus E^s$ is integrable.*

Having Lemmas 13, 14, 15 and 16 at hand, we are able to prove Theorem 17:

Proof of Theorem 17. By Lemma 16 we obtain a codimension one foliation on M tangent to $E^u \oplus E^s$. Consider a leaf L of this foliation such that $L \subseteq \overline{W^{uu}(p)} = K$. We will prove that $L = K$.

Since $W^{uu}(p) \subset L$, it is clear that $\overline{L} = K$, so we only need to check that L is a closed subset of M, or equivalently since M is a compact manifold, we only need to see that L is compact. Suppose that is not the case, i.e., that exists a point $x \in \overline{L} \setminus L$ and a pair of points u and v in L that are both near x on M but are very far from each other on L, as in Figure 4.2 bellow.

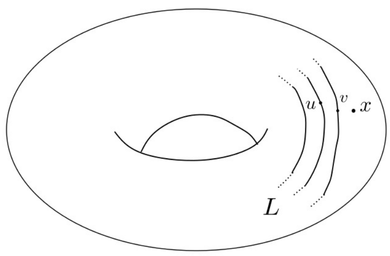

Figure 4.2: The points u and v on L and x on $\overline{L} \setminus L$.

Since $TL = E^u \oplus E^s$, the flow φ_t is transverse to L and hence we can refine our choice on u and v in such a way that $\varphi_t(u) = v$ for some $t > 0$ small (as in Figure 4.3). Hence, in one direction, we have $v \in L \subseteq K$; on the other hand, $v = \varphi_t(u) \in \varphi_t(K)$. So, $\varphi_t(K) \cap K \neq \emptyset$. This is a contradiction with Lemma 14. Therefore, $L = K$.

We have concluded that K is a closed leaf of a codimension one C^r-foliation, with $r \geq 1$, and therefore a compact codimension one C^1-submanifold of M. Last, we claim that $f\colon K \to K$ is an Anosov diffeomorphism.

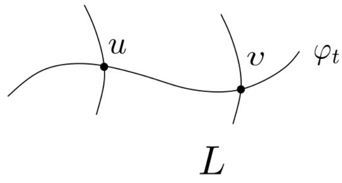

Figure 4.3: Refining the choice of u and v.

To see that, notice that f is the restriction of φ_t to the closed submanifold $K \subset M$ and, hence, f is a local diffeomorphism. It will follow from the proof of Lemma 14 that, if $\varphi_t(K) \cap K \neq \emptyset$, then $\varphi_t(K) = K$. Hence, f is surjective and this implies that $f\colon K \to K$ is a diffeomorphism. Additionally, since $T_xK = E^u(x) \oplus E^s(x)$, K is a hyperbolic set for f and so, f is an Anosov diffeomorphism which has suspension flow φ_t. $\square$

4.3 Proofs of the lemmas

Lemma 13

Lemma. *If $W^{uu}(p)$ is dense on M for all periodic point p then $W^{uu}(x)$ is dense on M for all $x \in M$.*

Given an arbitrary leaf $W \in \mathcal{F}^{uu}$, we need to show that $\overline{W} = M$. In other words, given an arbitrary point $x \in M$ and $\varepsilon > 0$ we need to show that every leaf $W \in \mathcal{F}^{uu}$ intersects $B_\varepsilon(x)$. To do so we shall use the compactness of M, the density of the periodic points of φ_t and the hypothesis that $W^{uu}(p)$ is dense on M for all periodic point p.

These hypothesis will guarantee a finite covering of M with balls centered on periodic points. An important step of the proof is to find an instant t such that each ball $B^{uu}_\varepsilon(p_i)$ of the covering intersects the fixed ball $B_\varepsilon(x)$ at the same time.

To do so, we need a beautiful result from Number Theory that generalizes the approximation of a real number by rationals to a more general setting, namely: given a $m \times n$ matrix $A \in \mathcal{M}_{m\times n}(\mathbb{R})$ with real entries and an arbitrary real number $\varepsilon > 0$, are there lattice points $k \in \mathbb{Z}^n$ and $h \in \mathbb{Z}^m$ such that the rows of Ak and h are arbitrarily close? As in the one-dimensional case, where A is a real number, the answer is yes. We reserve the Appendix B to answer this question and the consequence of this fact that is used to prove Lemma 13.

Proof. We need to show that fixed $x \in M$ and an ε-ball $B_\varepsilon(x)$ around it, every leaf W of $\mathcal{F}^{uu}$ intersects $B_\varepsilon(x)$. This is the path we follow.

For $r > 0$, we shall call $N_r(x)$ the following open set containing x:

$$N_r(x) = \bigcup_{y \in B^{uu}_r(x)} B^s_r(y).$$

It follows from Theorem 6 that, given such $r > 0$ there exists $\delta = \delta(r)$ such that $B_\delta(x) \subset N_r(x)$ for all $x \in M$. It is worth noticing that δ doesn't depend on $x \in M$: it follows from the Stable Manifold Theorem (Theorem 5 above) that the size of the embedded discs is uniform regarding to x. Now, fix $r > 0$ and let $\delta = \delta(r)$ be the associate δ.

Observe that, given $\delta > 0$ and the open cover $\bigcup_{y \in M} B_{\delta/2}(y)$ for M, there is a finite subcover $\bigcup_{i=1}^{k} B_{\delta/2}(y_i)$ of M. Now, since the set of periodic points is dense on M, in each of the balls $B_{\delta/2}(y_i)$ there is a periodic point p_i and hence,

$$B_{\delta/2}(y_i) \subseteq B_\delta(p_i),$$

and then $M = \bigcup_{i=1}^{k} B_\delta(p_i)$.

By hypothesis, we know that $W^{uu}(p_i)$ is dense for every $i = 1, \ldots, k$. In particular, this leads to the conclusion that $\{\varphi_t(B_r^{uu}(p_i)) \mid t \in \mathbb{R}\}$ is dense on M for each i and, hence, intersects $B_r(x)$ eventually. The next claim says that we can obtain an uniform time t such that the image of $B_r^{uu}(p_i)$ by the flow intersects $B_r(x)$ at the same t for all $i = 1, \ldots, k$.

Claim. *There exists $t > 0$ such that*

$$\varphi_t(B_r^{uu}(p_i)) \cap B_r(x) \neq \emptyset,$$

for all $i = 1, \ldots, k$.

Proof. By hypothesis, $\overline{W^{uu}(p_i)} = M$ for all $i = 1, \ldots, k$. So, there is $T > 0$ such that $B_T^{uu}(p_i) \cap B_r(x) \neq \emptyset$. By continuity of the foliation and since we are dealing with a finite number of points $\{p_1, \ldots, p_k\}$, exists $\lambda \in (0, 1)$ small enough such that, if t_i is the period of the point p_i by the flow, then:

$$|t| < \lambda t_i \implies \varphi_t(B_T^{uu}(p_i)) \cap B_r(x) \neq \emptyset, \tag{4.1}$$

for all $i = 1, \ldots, k$.

Now, consider the diffeomorphism $\varphi_{t_i}\colon M \to M$. Since the strong stable and strong unstable manifolds are invariant by it, we obtain:

$$\lim_{n \to \infty} \operatorname{diam}(\varphi_{nt_i}(B_r^{uu}(p_i))) = \infty.$$

Since $B_T^{uu}(p_i)$ is a bounded subset on $W^{uu}(p_i)$ that contains p_i and since mt_i is always a multiple of p_i's period for $m \in \mathbb{N}$, the set $\varphi_{mt_i}(B_r^{uu}(p_i))$ always contains p_i. Hence, there is a $n_i \in \mathbb{N}$ such that, if $m \geq n_i$, then

$$B_T^{uu}(p_i) \subseteq \varphi_{mt_i}(B_r^{uu}(p_i)). \tag{4.2}$$

We now need the approximation lemma from Number Theory that we mentioned before the proof and that we prove at Appendix B:

Lemma. *For all $\lambda, t_1, \ldots, t_N \in (0, +\infty)$ and $n_0 \in \mathbb{N}$, there are $n_1, \ldots, n_N \geq n_0$ and $t \in \mathbb{R}$ such that*

$$|n_i t_i - t| < \lambda t_i,$$

for all $i = 1, \ldots, N$.

The above lemma guarantees the existence of $t \in \mathbb{R}$ such that $|t - n_i t_i| < \lambda t_i$, for all $i = 1, \ldots, k$. Writing $t = n_i t_i + \varepsilon_i$, we obtain $|\varepsilon_i| < \lambda t_i$. By noticing that $\varphi_t(B_r^{uu}(p_i)) = \varphi_{\varepsilon_i}(\varphi_{n_i t_i}(B_r^{uu}(p_i)))$, it follows from (4.2) that $\varphi_{\varepsilon_i}(B_T^{uu}(p_i)) \subseteq \varphi_t(B_r^{uu}(p_i))$.

By the implication (4.1) above, it follows that $\varphi_{\varepsilon_i}(B_T^{uu}(p_i)) \cap B_r(x) \neq \emptyset$ and, therefore,

$$\varphi_t(B_r^{uu}(p_i)) \cap B_r(x) \neq \emptyset,$$

for all $i = 1, \ldots, k$. This proves the claim. $\square$

Back to the proof of Lemma 13, fix $t > 0$ given in the Claim we have just proven and consider $\varphi_{-t}(W) \in \mathcal{F}^{uu}$, where W is an arbitrary leaf of $\mathcal{F}^{uu}$. Since $M = \bigcup_{i=1}^{k} B_\delta(p_i)$, there is j ($1 \leq j \leq k$) such that $\varphi_{-t}(W) \cap B_\delta(p_j) \neq \emptyset$. Moreover, the Claim implies $\varphi_t(B_r^{uu}(p_j)) \cap B_r(x) \neq \emptyset$, i.e., there exists $q_j \in B_r^{uu}(p_j)$ such that $d(\varphi_t(q_j), x) < r$, for each j.

Also notice that, being $B_\delta(p_j) \subseteq N_r(p_j)$, we have $\varphi_{-t}(W) \cap N_r(p_j) \neq \emptyset$ and, by Theorem 6, there is an unique point $y \in \varphi_{-t}(W) \cap W_{loc}^s(q_j)$. Therefore, by applying the flow to q_j and y, the distance between these two points does not increase. So, if $d(q_j, y) < r$ it follows that $d(\varphi_t(q_j), \varphi_t(y)) < r$, and then:

$$d(x, \varphi_t(y)) \leq d(x, \varphi_t(q_j)) + d(\varphi_t(q_j), \varphi_t(y)) < 2r.$$

Since $\varphi_t(y) \in W$, we have shown that $W \cap B_{2r}(x) \neq \emptyset$, for an arbitrary $r > 0$, and this concludes the proof of Lemma 13. □

Lemma 14

Before proving Lemma 14, we need a general statement on flows acting on metric spaces:

Lemma. *Let (M, d) be a compact metric space and $\varphi\colon \mathbb{R} \times M \to M$ a continuous flow. If $K \subseteq M$ is compact, then*

$$\mathcal{U} := \bigcup_{a \leq t \leq b} \varphi_t(K)$$

is closed.

Proof. Let $p \in \overline{\mathcal{U}}$. There are $\varphi_{t_n}(x_n)$ with $t_n \in [a, b]$ and $x_n \in K$ such that

$$\varphi_{t_n}(x_n) \to p.$$

Since K is compact, there exists a subsequence x_{n_k} converging to some $x \in K$ as $k \to +\infty$. Again by compactness, there is some subsequence $t_{n_{k_j}}$, now of $(t_{n_k})_k$, which converges to some $t \in [a, b]$.

By the flow's continuity, the sequence $\varphi_{t_{n_{k_j}}}(x_{n_{k_j}})$ converges to $\varphi_t(x)$. Since it also converges to p, we conclude that $\varphi_t(x) = p \in \mathcal{U}$, showing that $\overline{\mathcal{U}} \subseteq \mathcal{U}$, and hence that $\mathcal{U}$ is a closed set. □

We now are able to prove Lemma 14:

Lemma. *If $p \in \operatorname{Per}(\varphi)$ is such that $\overline{W^{uu}(p)} \neq M$, then:*

- *M is a fiber bundle;*
- *with base space S^1;*
- *with fiber $K = \overline{W^{uu}(p)}$;*
- *and φ_t is the suspension flow of an homeomorphism $f\colon K \to K$.*

Proof. Let $\tau \in \mathbb{R}$ be the period of the point p through φ. Since $W^{uu}(p)$ is invariant by φ_τ, we have:

$$\begin{aligned}\bigcup_{t \in \mathbb{R}} \varphi_t(W^{uu}(p)) &\subseteq \bigcup_{t \in \mathbb{R}} \varphi_t\left(\overline{W^{uu}(p)}\right) \\ &= \bigcup_{0 \leq t \leq \tau} \varphi_t\left(\overline{W^{uu}(p)}\right).\end{aligned}$$

Since, by the previous Lemma, $\bigcup_{0\leq t\leq\tau}\varphi_t\left(\overline{W^{uu}(p)}\right)$ is closed and, by Theorem 16, $W^u(p)$ is dense on M, we obtain:

$$M = \bigcup_{t_0\leq t\leq\tau}\varphi_t\left(\overline{W^{uu}(p)}\right).$$

Now, let $K\subset\overline{W^{uu}(p)}$ be a **non-empty** minimal set with respect to the following conditions:

(i) K is a closed subset of M;

(ii) K is $\mathcal{F}^{uu}-$saturated;

(iii) K is invariant by the φ_τ, i.e., $\varphi_\tau(K)=K$.

The existence of a non-empty set $K\subset\overline{W^{uu}(p)}$ satisfying properties (i), (ii), and (iii) above, is guaranteed by Zorn's Lemma. Now define $\mathcal{U}\coloneqq\bigcup_{t\in[0,\tau]}\varphi_t(K)$. We claim that $\mathcal{U}=M$.

In order to see that we notice that K is $\varphi_\tau-$invariant, $\mathcal{F}^u-$saturated (since K is $\mathcal{F}^{uu}-$saturated) and closed (by compactness of M and by the flow's continuity). Moreover, $\mathcal{U}$ being non-empty and $\mathcal{F}^u$ being a minimal foliation (by Theorem 16), we must have $\mathcal{U}=M$.

At this point, we know that $\mathcal{U}\coloneqq\bigcup_{t\in[0,\tau]}\varphi_t(K)$. However, to prove Lemma 14, we still need to put some effort on checking two important assertions:

(a) the union in $\mathcal{U}$ is, in fact, disjoint:

$$\bigcup_{t\in[0,\tau]}\varphi_t(K)=\bigsqcup_{t\in[0,\tau]}\varphi_t(K);$$

(b) the minimal set K is actually $\overline{W^{uu}(p)}$.

The statement (b) is pretty direct: since $p\in K$ and since K is $\mathcal{F}^u-$saturated, we must have $W^{uu}(p)\subseteq K$. Therefore, since $\overline{W^{uu}(p)}$ satisfies (i), (ii) and (iii), we conclude by minimality, that we must have $\overline{W^{uu}(p)}=K$.

To prove (a), let $t\in[0,\tau]$ be such that

$$K_t\coloneqq K\cap\varphi_t(K)\neq\emptyset.$$

Since $\varphi_t(K)$ is closed, $\varphi_\tau-$invariant, and $\mathcal{F}^{uu}-$saturated (by the invariance of the strong unstable foliation by φ_t), minimality implies $K\subseteq K_t$. In particular, this leads to $K\subseteq\varphi_t(K)$ and hence $\varphi_{-t}(K)\subseteq K$.

Naturally, the set $\varphi_{-t}(K)$ also satisfies the properties (i), (ii) and (iii), and by minimality we have $K\subseteq\varphi_{-t}(K)$ and hence:

$$K=\varphi_t(K).$$

So now, we have the information that $K=\varphi_t(K)$ is equivalent to $K\cap\varphi_t(K)\neq\emptyset$. With expectations that this equivalence leads to something, define:

$$\begin{aligned}\mathcal{G}&=\{s\in\mathbb{R}\mid K\cap\varphi_s(K)\neq\emptyset\}\\&=\{s\in\mathbb{R}\mid K=\varphi_s(K)\}.\end{aligned}$$

This set is clearly non-empty since $0 \in \mathcal{G}$ and, in fact, it is an additive subgroup of $(\mathbb{R}, +)$. Indeed, if $a, b \in \mathcal{G}$, then

$$\varphi_{a+b}(K) = \varphi_a(\varphi_b(K)) = \varphi_a(K) = K$$

and

$$\varphi_{-a}(K) = \varphi_{-a}(\varphi_a(K)) = K.$$

Consider now $s_0 = \inf(\mathcal{G} \cap \mathbb{R}_{>0})$. We claim that $s_0 = \tau$. To prove it, one should ask what cannot occur: the only thing that could happen to make the claim false is that $s_0 = 0$; otherwise, if $s_0 > 0$ then $s_0 = \tau$, since τ is the period of $p \in K$.

So, suppose by contradiction that $s_0 = 0$. In particular, assume that $\mathcal{G}$ accumulates at 0. Being an additive subgroup of $\mathbb{R}$, this implies $\overline{\mathcal{G}} = \mathbb{R}$. We are going to show that actually, in this case, $\mathcal{G} = \mathbb{R}$. To do so, consider $\tilde{t} \in \mathbb{R}$ and a sequence $(t_n)_n \subseteq \mathcal{G}$ with $t_n \to \tilde{t}$. Since $t_n \in \mathcal{G}$ for every $n \geq 1$, we have

$$\varphi_{t_n}(K) = K,$$

for all $n \geq 1$. Since the flow is continuous and K is a compact set (it is closed on M), we conclude that:

$$d_H(\varphi_{t_n}(K), \varphi_{\tilde{t}}(K)) \to 0$$

as $n \to \infty$ (here d_H is the Hausdorff's distance on M). So, $\varphi_{\tilde{t}}(K) = K$ and then $\mathcal{G} = \mathbb{R}$.

By definition, we have just proven that $\varphi_t(K) = K$ for all $t \in \mathbb{R}$ and, in particular, that

$$K = \bigcup_{t \in \mathbb{R}} \varphi_t(K).$$

Since $\varphi_t(K)$ is $\mathcal{F}^u-$saturated and since the union of saturated sets is still saturated, we conclude that along with non-empty and closed, K is also $\mathcal{F}^u-$saturated. So, applying Theorem 16 once more, we conclude that $K = M$.

However, this cannot occur! Since $K \subset \overline{W^{uu}(p)}$, if $K = M$, we would conclude that $\overline{W^{uu}(p)} = M$, a contradiction with the hypothesis of this lemma. So, $0 < s_0$ and as discussed above, $s_0 = \tau$. This shows that

$$\bigcup_{t \in \mathbb{R}} \varphi_t(K) = \bigcup_{t \in [0,\tau]} \varphi_t(K) = \bigsqcup_{t \in [0,\tau]} \varphi_t(K).$$

To conclude the proof of this lemma, we only need to observe carefully what we have done: first, by these observations (a) and (b) we have

$$M = \bigsqcup_{t \in [0,\tau]} \varphi_t(K),$$

since $\mathcal{U} = \bigcup_{t \in [0,\tau]} \varphi_t(K)$ is a non-empty, closed, and $\mathcal{F}^u-$invariant set. Moreover, if we consider the projection from M to S^1 given by the map which associates $\varphi_t(K) \mapsto t \pmod r$ and the maps $h = \varphi_\tau|_K$, to get the fibration and homeomorphism claimed at the lemma's statement. $\square$

Lemma 15

To give a complete proof of Lemmas 15 and 16, and hence to establish Theorem 17, we need to define what are **jointly integrable foliations**.

Let $N = G(B^s_\delta(x) \times B^{uu}_\delta(x))$ be a product neighborhood of a point $x \in M$ as in Theorem 6. If y and z are at the same strong unstable manifold on N, then there exists $\delta' > 0$ such that the $F_{y,z} \colon B^s_{\delta'}(y) \to B^s_\delta(z)$ given by the projections onto the strong unstable manifolds is well defined.

Definition. *The foliations $\mathcal{F}^{uu}$ and $\mathcal{F}^{ss}$ are said to be* ***jointly integrable on*** *N if, for y, z and δ' as above, we have:*

$$F_{y,z}\left(W^{ss}\left(u\right)\cap B^{s}_{\delta'}\left(y\right)\right)\subseteq W^{ss}\left(F_{y,z}\left(u\right)\right)\cap B^{s}_{\delta}\left(z\right)$$

where $u\in B^{s}_{\delta'}\left(y\right)$. Moreover, we say that $\mathcal{F}^{uu}$ and $\mathcal{F}^{ss}$ are ***jointly integrable*** *if each point on M belongs at a product neighborhood N where both $\mathcal{F}^{uu}$ and $\mathcal{F}^{ss}$ are* ***jointly integrable****. In this last case we also call the bundles E^{u} and E^{s}* ***jointly integrable****.*

At this point we should observe that the knowledge of a foliation to be jointly integrable gives extra information on the holonomy $F_{y,z}$. Whenever $u\in B^{s}_{\delta'}(y)$ and we consider the image of the strong stable leaf through u by the holonomy, we obtain a continuous curve. At first, this curve do not need to be at the strong stable leaf through $F_{y,z}(u)$. To ask that the foliation is jointly integrable is precisely to ask, in fact, this curve is entirely contained on the strong stable leaf through $F_{y,z}(u)$.

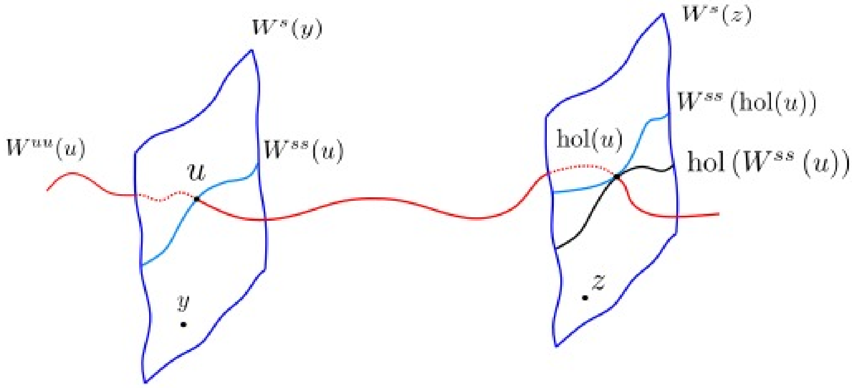

Figure 4.4: $F_{yz}(W^{ss}(u))\neq W^{ss}\left(F_{yz}\left(u\right)\right)$, i.e., non-jointly integrable foliations.

We now are able to prove Lemma 15, which claimed:

Lemma. *If some strong unstable (stable) manifold is not dense on M, then the foliations $\mathcal{F}^{uu}$ and $\mathcal{F}^{ss}$ are jointly integrable.*

Proof. By Corollary 4 there is a point $p\in\mathrm{Per}\left(\varphi\right)$ such that $\overline{W^{uu}(p)}\neq M$, and therefore we are in conditions to apply Lemma 14 and then think of M as a fiber bundle with fiber $K=\overline{W^{uu}(p)}$.

That K is $\mathcal{F}^{uu}-$saturated it is clear. Here we are going to show that K is also $\mathcal{F}^{ss}-$saturated and then show that this two conditions happening simultaneously implies jointly integrability.

To see that K is indeed $\mathcal{F}^{ss}-$saturated, we proceed by contradiction: suppose that is not the case, i.e., that there exists a point $v\in K$ such that $\mathcal{F}^{ss}(v)\nsubseteq K$. Therefore exists another fiber $K'=\varphi_{t}(K)$ for some $t\in(0,\tau)$, such that

$$\mathcal{F}^{ss}(v)\cap K'\neq\emptyset.$$

Pick a point $u\in\mathcal{F}^{ss}(v)\cap K'$, as in the Figure 4.5 bellow.

Now, we use the homeomorphism $h\colon K\to K$ defined in Lemma 14. Since $h=\varphi_{\tau}|_{K}$, the map h sends fiber to fiber we have, in one hand:

$$d\left(h^{n}(u),h^{n}(v)\right)\geq d_{H}\left(K',K\right)>0.$$

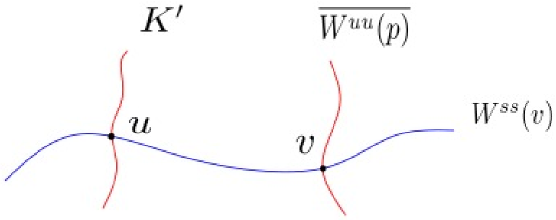

Figure 4.5: $u \in \mathcal{F}^{ss}(v) \cap K'$

However, since $u \in W^{ss}(v)$ we must have $d\left(h^n(u), h^n(v)\right) \to 0$ as $n \to \infty$. This provides a contradiction and shows that K is also $\mathcal{F}^{ss}-$saturated.

Now we wish to prove that the fact that K is both $\mathcal{F}^{uu}-$saturated and $\mathcal{F}^{ss}-$saturated, implies that the foliations $\mathcal{F}^{uu}$ and $\mathcal{F}^{ss}$ are jointly integrable. To do that suppose, once again, the opposite: that they are not jointly integrable. Then there are $y, z \in M$ such that $F_{yz}(W^{ss}(y)) \neq W^{ss}\left(F_{yz}\left(z\right)\right)$, for $z \in W^{uu}_{loc}(y)$.

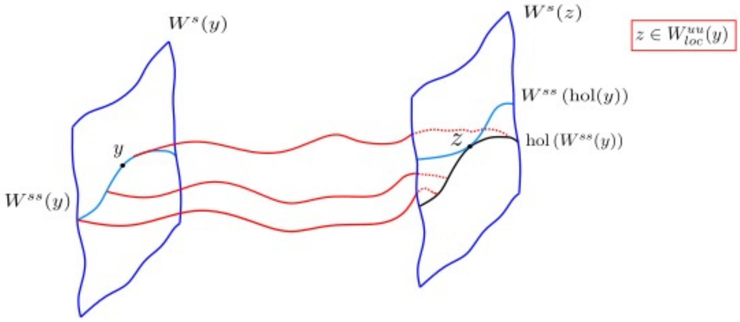

Figure 4.6: The failure of jointly integrability.

Since K is simultaneously $\mathcal{F}^{uu}$ and $\mathcal{F}^{ss}$ saturated, the sets $W^{uu}_{loc}(y)$ and $F_{yz}(W^{uu}_{loc}(y))$ are both contained in the same fiber, since F_{yz} is the holonomy map and just moves $W^{uu}_{loc}(y)$ along $\mathcal{F}^{uu}$.

Now, for small enough t, if we apply the flow to $F_{yz}(W^{uu}_{loc}(y))$, we must have

$$\varphi_t\left(F_{yz}(W^{uu}_{loc}(y))\right) \cap W^{ss}\left(F_{yz}(y)\right) \neq \emptyset,$$

(see Figure 4.7) and hence $\varphi_t(K) \cap K \neq \emptyset$, a contradiction. Hence, $\mathcal{F}^{uu}$ and $\mathcal{F}^{ss}$ are jointly integrable.

□

Lemma 16

Finally, to prove Lemma 16, we remember Definition 4, where we define what means a foliation to be integrable:

Definition. *Let M be a smooth manifold and let $E \subseteq TM$ be a continuous subbundle of the tangent bundle. We call E* ***integrable*** *if it is the tangent bundle of a $C^{0,1}$-foliation, i.e., a C^0 foliation with C^1 leafs.*

Lemma. *The foliations $\mathcal{F}^{uu}$ and $\mathcal{F}^{ss}$ are jointly integrable if and only if the splitting $E^u \oplus E^s$ is integrable.*

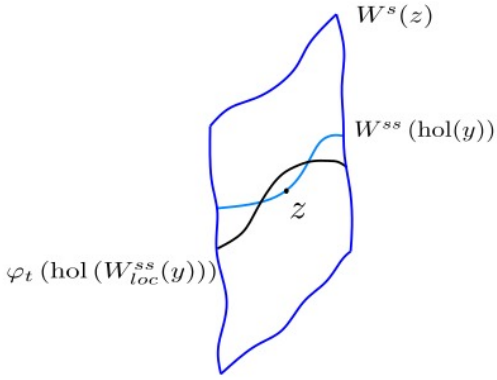

Figure 4.7: Applying the flow to $F_{yz}(W^{uu}_{loc}(y))$ for small enough t.

Proof. If $E^u \oplus E^s$ is integrable, is quite straightforward to see that $\mathcal{F}^{uu}$ and $\mathcal{F}^{ss}$ are jointly integrable. Indeed, call $T(\mathcal{F}^{us}) = E^u \oplus E^s$. Since the holonomy of $W^{ss}(x)$ trough $\mathcal{F}^{uu}$ is tangent to E^s, it follows that $F_{xz}(W^{ss}(x)) = W^{ss}(F_{xz}(x))$, for $z \in W^{uu}(x)$. Since x and z were taken arbitrarily, it follows that $\mathcal{F}^{uu}$ and $\mathcal{F}^{ss}$ are jointly integrable.

Reciprocally, suppose the foliations are jointly integrable. Then, through each point $x \in M$ there exists an embedded C^1-submanifold of M, say L_x, such that its tangent space $T_x(L_x)$ is $E^u_x \oplus E^s_x$. What we need to do is to check that this collection of C^1-submanifolds is, in fact, a foliation.

To do so, call by $\mathcal{F}$ such a collection. We are going to construct a foliated chart to $\mathcal{F}$. To do so, through each point $x_0 \in M$, consider η a C^1-embedding of an open disk $B(x_0)$ of L_{x_0} around x_0 to $\mathbb{R}^{n-1}$.

Applying the flow to $B(x_0)$ for small enough $\delta > 0$ we obtain an open set $\mathcal{U} := \bigcup_{|t|<\delta} \varphi_t(B(x_0))$ in a way that the flow map $B(x_0) \times (-\delta, \delta) \to \mathcal{U}$ defined by $(x,t) \mapsto \varphi_t(x)$ is an diffeomorphism onto its image.

Moreover, since $d\varphi_t(E^u_x \oplus E^s_x) = E^u_{\varphi_t(x)} \oplus E^s_{\varphi_t(x)}$, the flow sends "leafs"[2] in $\mathcal{F}$ to "leafs" in $\mathcal{F}$. So, defining the map $\psi \colon \mathcal{U} \to \mathbb{R}^{n-1} \times \mathbb{R}$ by $\psi(\varphi_t(x)) = (\eta(x), t)$, we obtain a C^1-foliated chart around x_0 in M. Now, varying $x_0 \in M$, gives a C^1-foliation $\mathcal{F}$ tangent to $E^u \oplus E^s$. Thus, is integrable. □

4.4 Horocycle flows are minimal

Theorem 15 shows that a transitive Anosov flow on a compact Riemannian manifold is either minimal or a suspension flow. This has a very nice implication in the case of geodesic flows:

Theorem 18. *The horocycle flow associated to a geodesic flow $g_t \colon T^1S \to T^1S$ on the unit tangent bundle a surface of constant negative curvature is minimal.*

Proof. Indeed, since $W^{uu}(x) = \{h_s(x) \mid s \in \mathbb{R}\}$ and since g_t cannot be a suspension flow, Theorem 15 implies that $W^{uu}(x)$ is dense for every $x \in M$ or, equivalently, the orbit of x by h_s, i.e., $\mathcal{O}_h(x) = \{h_s(x) \mid s \in \mathbb{R}\}$ is dense for every $x \in M$. This shows that h_s is a minimal flow. □

[2]At this point we don't know already that the elements of $\mathcal{F}$ are leafs of a foliation.

CHAPTER 5

Unique ergodicity of the horocycle Flow

We now establish an important property of the horocycle flow: ergodicity. In order to do so, we need to analyze finer structure of measures on M that are invariant under a flow $\psi_s\colon M \to M$ that is transverse to a foliation $\mathcal{F}$ on M.

Recall that, if $\varphi_t\colon M \to M$ is an Anosov flow on a closed Riemannian manifold M, we have a local product structure on M. In particular, there are foliated charts on which the plaques are unstable manifolds and charts on which the plaques are strong stable manifolds, transverse to the previous one.

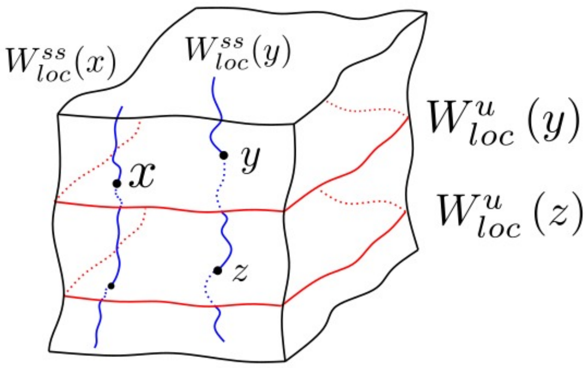

Figure 5.1: A product neighborhood with transverse leafs from $\mathcal{F}^u$ and $\mathcal{F}^{ss}$.

Throughout this chapter we will study measures that respect this local product structure.

Definition 23. *We say a measure μ has a* ***local product structure*** *if, in a small enough foliated chart $V = L \times I \subseteq M$, with $L \in \mathcal{F}$, the measure μ can be disintegrated, up to renormalization of the measures, as*

$$\int_V f\, d\mu = \int_I \int_L f(x,s)\, d\nu_s\, ds,$$

for all $f \in C^0(M)$. Here, ds is the Lebesgue measure on I and the ν_s are probability measures on L that vary measurably on s.

With that definition at hand we can state the main theorem of this chapter:

Theorem 19. *Let $g_t\colon M \to M$ be an Anosov flow on a compact Riemannian manifold such that the stable foliation $W^{ss}(x)$ has constant dimension equal 1. Suppose, moreover, that the stable foliation W^{ss} is parametrized by a continuous flow $h_s\colon M \to M$, that the volume measure μ on M is invariant under both flows g_t and h_s, that μ has local product structure and that*

$$g_t \circ h_s = h_{se^{-t}} \circ g_t,$$

for every $t, s \in \mathbb{R}$.

Then, if h_s has a dense orbit, it is uniquely ergodic.

We follow a very elegant proof by Yves Coudène [Cou09]. Surely the flow g_t mimics the geodesic flow, which has all the properties on the theorem, as we shall see bellow.

The proof presented here follows from a series of lemmas. Being a theorem on ergodicity, we recall the definition Birkhoff sums associated, now associated with the flow h_s:

$$S_t(f)(x) = \int_0^t f\left(h_s\left(x\right)\right) ds,$$

where $f\colon X \to \mathbb{R}$ is a continuous function. Through this chapter, unless stated otherwise, the expression $S_t(f)(x)$ will always refer to the Birkhoff sum associated with the flow h_s.

To make the proof of the main theorem of this chapter clearer, we restate Proposition 7 from Chapter 2 as a lemma, and stated specifically to the flow h_s:

Lemma 17. *If, for every sequence $(t_k)_k$ with $t_k \to +\infty$ such that the uniform limit*

$$\lim_{k\to+\infty} \frac{1}{t_k} S_{t_k}(f)$$

exists, the limit function is constant, then h_s is uniquely ergodic.

So, the main work is to show that, with the hypotheses of the theorem, we can achieve the hypothesis of this Lemma 17. Lemma 17 will imply unique ergodicity of h_s.

To see that the hypotheses of Theorem 19 implies the hypothesis of Lemma 17 we use following chain of lemmas:

Lemma 18. *Let $\mathcal{M} = \{M_t(f)\}_{t>0}$ be a family of functions defined by*

$$M_t(f)(x) = \int_0^1 f\left(g_{-\ln t}\left(h_s(x)\right)\right)\, ds.$$

Then $\mathcal{M}$ is equicontinuous and each of its accumulation point on $C^0(X)$ is a constant function.

Lemma 19. *With respect to $S_t(f)$ we have:*

(1) *the family $\mathcal{S} = \left\{\frac{1}{t}S_t(f)(x)\right\}_{t>0}$ has compact closure on $C^0(X)$;*

(2) *each accumulation point of $\mathcal{S}$ is constant.*

As we shall see in a while, the hypothesis on Theorem 19 prove Lemma 18 and also we prove that Lemma 18 implies Lemma 19. Nevertheless, with the lemmas at hand, it is easy to prove Theorem 19:

Proof. From Lemma 19, we know that if $(t_k)_k$ is a sequence with $t_k \to +\infty$, then

$$\lim_{k\to+\infty} \frac{1}{t_k} S_{t_k}(f)$$

has a limit (since $\mathcal{S}$ has compact closure on $C^0(X)$) and this limit is constant. Now, unique ergodicity follow from Lemma 17. □

5.1 Proving the three lemmas

The three lemmas follow the logical chain:

$$\begin{array}{c}\boxed{\text{Lemma 18}}\\ \downarrow \\ \boxed{\text{Lemma 19}}\\ \downarrow \\ \boxed{\text{Lemma 17}}\end{array}$$

First, we use the hypothesis of Theorem 19 to check Lemma 18: namely we strongly use the hyperbolicity of g_t and the relation between g_t and h_s and the invariance of the Liouville measure μ under both flows.

For the proof of Lemma 19 we use Lemma 18, the relation between g_t and h_s given by the equation $g_t \circ h_s = h_{se^{-t}} \circ g_t$, and the existence of a dense orbit by h_s.

Lemma 18

Lemma 18 is where the heart of the proof relies. To prove it we need to break it into two pieces: first we show that the family of functions $\mathcal{M} = \{M_t(f)\}_{t>0}$, where

$$M_t(f)(x) = \int_0^1 f\left(g_{-\ln t}\left(h_s(x)\right)\right)\, ds,$$

is equicontinuous and then we prove that all its accumulation points on $C^0(X)$ are constant functions.

Before proving equicontinuity, we need several lemmas that explore the hyperbolicity of the flow g_t, the relation between g_t and h_s and finally the fact that the measure μ is invariant under both flows. One example is the following Lemma 20, where we use the fact the we are asking the flows to be Anosov, hence of regularity at least C^1:

Lemma 20. *Locally, the Liouville measure μ satisfy the local product structure as in Definition 23, where $L \in \mathcal{F}^u$ is the center-unstable leaf and $I \in W^s$ is a stable leaf which is parametrized by h_s.*

Proof. This Lemma is a direct consequence of the characterization of the measure μ done in Proposition 4.1.4, on p. 68 of [Alv13].

□

For the next Lemma, set $V_{x_0} \subseteq M$ to be a open subset of the product neighborhood $N(x_0)$ of the point $x_0 \in M$, i.e., we may think as $N(x_0)$ as a subset of $W^u(x_0) \times W^{ss}(x_0)$.

Since the flow h_s parametrizes the one dimensional submanifold $W^{ss}(x_0)$, any point $x \in N(x_0)$ can be described as a pair $(y(x), s(x))$, where $y(x) \in W^u(x_0)$ is the projection of x onto $W^u(x_0)$ and $s(x)$ represents the projection of x on $W^{ss}(x_0)$. On that setting we define, for every point in $N(x_0)$, the neighborhood V_x of x defined as

$$V_x = B(y(x), \delta) \times (s(x), s(h_1(x))),$$

where $B(y(x),\delta)$ is a ball in $W^u(x_0)$ and $(s(x)),s(h_1(x)))$ an open segment contained in the one-dimension submanifold $W^{ss}(x_0)$.

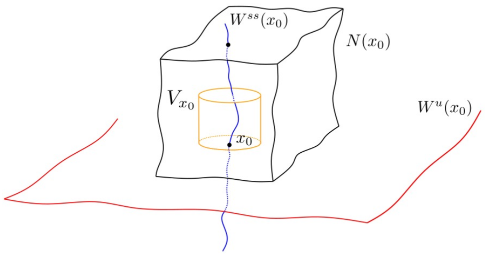

Figure 5.2: The neighborhood V_{x_0}.

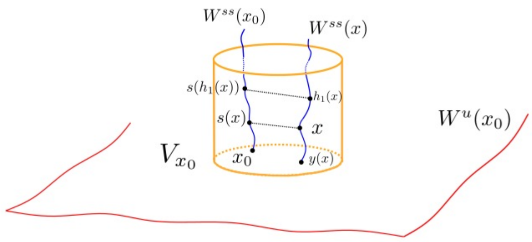

Figure 5.3: Local coordinates for $x \in V_{x_0}$.

Lemma 21. *The family of functions* $\frac{1}{\mu(V_x)}\chi_{V_x}$ *converges to* $\frac{1}{\mu(V_{x_0})}\chi_{V_{x_0}}$ *in* $L^2(\mu)$*, as* $x \to x_0$*.*

Before proving this Lemma 21, we need more two lemmas:

Lemma 22. *Let* (X,Σ,μ) *a probability space and* $(\psi_n)_n$ *a sequence of measurable functions* $\psi_n \colon X \to \mathbb{R}$*, uniformly bounded, i.e., there is* $C \geq 0$ *such that* $\|\psi_n\|_\infty \leq C$ *for all* $n \in \mathbb{N}$*. Suppose* ψ_n *converge pointwise to a function* $\psi \colon X \to \mathbb{R}$*. Then,* ψ *is integrable and*

$$\lim_{n\to+\infty} \int_X \psi_n \, d\mu = \int_X \psi \, d\mu.$$

Proof. Define the function $f(x) = C$ for all $x \in X$, so that

$$|\psi_n(x)| \leq f(x),$$

for all $x \in X$ and all $n \in \mathbb{N}$. Since $\mu(X) = 1$, we have that $\int_X f \, d\mu = C \cdot \mu(X) = C < +\infty$. Hence, by the Dominated Convergence Theorem, we know that ψ is integrable and that

$$\lim_n \int_X \psi_n \, d\mu = \int_X \lim_n \psi_n \, d\mu = \int_X \psi \, d\mu.$$

□

Lemma 23. *Let* (X,d) *be a compact metric space and* μ *a Borel probability measure on* X *with full support. Then, for every* $x \in X$ *and* $r > 0$ *there is some* $0 < \delta < r$ *such that* $\mu\left(\partial B(x,\delta)\right) = 0$*.*

Proof. Suppose the contrary, i.e., that there exists some $x_0 \in X$ and some $r_0 > 0$ such that for all $0 < \delta < r_0$ we have $\mu\left(\partial B(x,\delta)\right) > 0$.

We claim that there exists some constant $C > 0$ and a sequence $(r_n)_n$ with $r_n \in (0, r_0)$ such that $r_{n+1} < r_n$ and

$$\mu\left(\partial B(x_0, r_n)\right) > C,$$

for all $n \in \mathbb{N}$.

To prove the claim, notice that for each $\delta \in (0, r_0)$ there exists some $n = n(\delta) \in \mathbb{N}$ such that $\mu\left(\partial B(x_0,\delta)\right) > \frac{1}{n}$. So, if we define

$$S_n = \left\{ \delta \in (0, r_0) \mid \mu\left(\partial B(x_0,\delta)\right) > \frac{1}{n} \right\}$$

we have $(0, r_0) = \bigcup_{n=1}^{+\infty} S_n$ and $S_n \subseteq S_{n+1}$ for all $n \in \mathbb{N}$.

Since $(0, r_0)$ is uncountable, there is some $n_0 \in \mathbb{N}$ such that S_{n_0} must also be uncountable. Now, take a monotone decreasing sequence on S_{n_0}, say $(r_n)_n$, and take $C = \frac{1}{n_0}$. This proves the claim.

Back to the proof of the main statement, notice that

$$\bigcup_{n=1}^{\infty} \partial B(x_0, r_n) = \bigsqcup_{n=1}^{\infty} \partial B(x_0, r_n) \subseteq \overline{B(x_0, r_0)}$$

and hence:

$$\begin{aligned} 1 \geq \mu\left(\overline{B(x_0, r_0)}\right) &\geq \mu\left(\bigcup_{n=1}^{\infty} \partial B(x_0, r_n)\right) \\ &= \mu\left(\bigsqcup_{n=1}^{\infty} \partial B(x_0, r_n)\right) \\ &= \sum_{n=1}^{\infty} \mu\left(\partial B(x_0, r_n)\right) \\ &\geq n \cdot C, \end{aligned}$$

for every $n \in \mathbb{N}$, (since $r_n \in S_m$ for every $n \in \mathbb{N}$). This absurd concludes the proof. $\square$

Proof of Lemma 21. The idea is to prove that $\frac{1}{\mu(V_x)}\chi_{V_x}$ converges to $\frac{1}{\mu(V_{x_0})}\chi_{V_{x_0}}$ almost everywhere and then use Lemma 22 to obtain the L^2-convergence.

To do so, we first show convergence for the characteristic functions χ_{V_x}. For, choose $\delta > 0$ such that $\mu(\partial V_{x_0}) = 0$ (which is possible by Lemma 23). Notice that, for all $z \in M \setminus \partial V_{x_0}$, we have

$$\lim_{x \to x_0} \chi_{V_x}(z) = \chi_{V_{x_0}}(z).$$

Indeed, there are two options: first $z \in M \setminus \overline{V_{x_0}}$. In this case there is an open neighborhood $\mathcal{O}_{x_0}$ of x_0 such that, for all $x \in \mathcal{O}_{x_0}$ we have $\chi_{V_x}(z) = \chi_{V_{x_0}}(z) = 0$. On the other hand, if $z \in V_{x_0}$, we also have an open neighborhood $\mathcal{O}_{x_0}$ of x_0 such that for all $x \in \mathcal{O}_{x_0}$, the equality $\chi_{V_x}(z) = \chi_{V_{x_0}}(z) = 1$ hold. Hence, for almost every $z \in M$, $\chi_{V_x}(z) \to \chi_{V_{x_0}}(z)$ as $x \to x_0$.

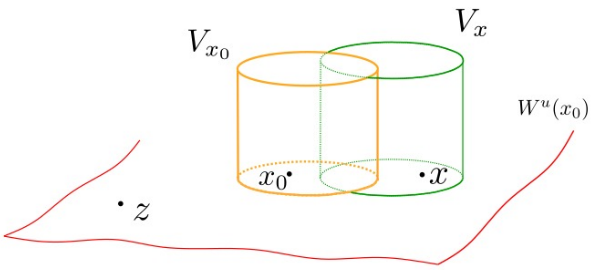

Figure 5.4: $z \in M \setminus \overline{V_{x_0}}$.

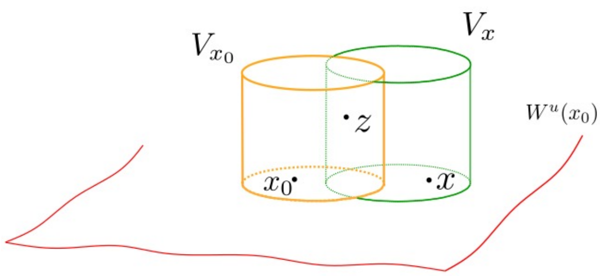

Figure 5.5: $z \in V_{x_0}$.

The previous statements can be made formal. To do so, we use the neighborhood

$$\begin{aligned} V_{x_0} &= B(y(x_0), \delta) \times (s(x_0), s(h_1(x_0))) \\ &= B(x_0, \delta) \times (0, 1), \end{aligned}$$

where again $B(x_0, \delta)$ is a ball in $W^u(x_0)$. By continuity of h_s we have that, when $x \to x_0$, the coordinates $y(x)$, $s(x)$ and $s(h_1(x))$ tend to $y(x_0) = x_0$, $s(x_0) = 0$ and $s(h_1(x_0)) = 1$, respectively.

This local understanding allow us to give a more precise assertion to what happen to the maps $\chi_{V_{x_0}}$ and χ_{V_x} applied to $z \in M \setminus \partial V_{x_0}$ when $x \to x_0$. Indeed, the previous paragraph implies, in particular, that $\overline{V_x}$ converges to $\overline{V_{x_0}}$ on the Hausdorff topology, and then $z \in V_{x_0}$ implies $\chi_{V_x}(z) \to 1 = \chi_{V_{x_0}}$ and $z \in M \setminus \overline{V_{x_0}}$ implies $\chi_{V_x}(z) \to 0 = \chi_{V_{x_0}}$, as claimed.

Now, let $(x_n)_n$ to be any sequence on $M \setminus \partial V_{x_0}$ with $x_n \to x_0$. We will show that

$$\|\chi_{V_{x_n}} - \chi_{V_{x_0}}\|_2 \to 0,$$

as $n \to +\infty$.

Indeed, $\chi_{V_{x_n}}(z) \to \chi_{V_{x_0}}(z)$ pointwise for $\mu-$a.e. z and for all $n \in \mathbb{N}$,

$$\|\chi_{V_{x_n}}\|_\infty \leq 1 \text{ and } \|\chi_{V_{x_0}}\|_\infty \leq 1.$$

By calling $\psi_n(z) = |\chi_{V_{x_n}}(z) - \chi_{V_{x_0}}(z)|^2$ we have, from Lemma 22,

$$\begin{aligned}
\lim_{n\to+\infty} \int_X |\chi_{V_{x_n}}(z) - \chi_{V_{x_0}}(z)|^2 \, d\mu &= \lim_{n\to+\infty} \int_X g_n \, d\mu \\
&= \int_X \lim_{n\to+\infty} g_n \, d\mu \\
&= \int_X 0 \, d\mu = 0,
\end{aligned}$$

i.e., we have shown that $\chi_{V_{x_n}} \xrightarrow[n\to+\infty]{L^2(\mu)} \chi_{V_{x_0}}$.

Since $\chi_{V_{x_n}}(z) - \chi_{V_{x_0}}(z) \in \{0, 1\}$ for all $z \in M$, we have that $|\chi_{V_{x_n}}(z) - \chi_{V_{x_0}}(z)| = |\chi_{V_{x_n}}(z) - \chi_{V_{x_0}}(z)|^2$, and then, $L^2(\mu)$-convergence implies $L^1(\mu)$-convergence. Therefore,

$$\begin{aligned}
|\mu(V_{x_n}) - \mu(V_{x_0})| &= \left| \int_X \chi_{V_{x_n}} \, d\mu - \int_X \chi_{V_{x_0}} \, d\mu \right| \\
&= \left| \int_X \chi_{V_{x_n}} - \chi_{V_{x_0}} \, d\mu \right| \\
&\leq \int_X |\chi_{V_{x_n}} - \chi_{V_{x_0}}| \, d\mu \to 0,
\end{aligned}$$

as $n \to +\infty$. This shows that $\mu(V_x) \to \mu(V_{x_0})$ as $x \to x_0$.

Hence, we have that the family of functions $\frac{1}{\mu(V_x)}\chi_{V_x}(z)$ converges to the function $\frac{1}{\mu(V_{x_0})}\chi_{V_{x_0}}(z)$ for $\mu-$almost every $z \in M$. Since $\mu(V_x) \to \mu(V_{x_0})$, we know that both $\frac{1}{\mu(V_x)}\chi_{V_x}$ and $\frac{1}{\mu(V_{x_0})}\chi_{V_{x_0}}$ are uniformly bounded. Applying one more time Lemma 22, we finally conclude $L^2(\mu)-$convergence:

$$\left\| \frac{1}{\mu(V_x)}\chi_{V_x} - \frac{1}{\mu(V_{x_0})}\chi_{V_{x_0}} \right\|_2 \to 0,$$

as $x \to x_0$.

$\square$

To finally prove Lemma 18 we present a relation between the $M_t(f)(x) = \int_0^1 f\left(g_{-\ln t}\left(h_s(x)\right)\right) \, ds$ and the Birkhoff sums of the horocycle flow $S_t(f)(x) = \int_0^t f\left(\varphi_s(x)\right) \, ds$.

Lemma 24. *For every $t \in \mathbb{R}$, every continuous function $f : X \to \mathbb{R}$, and $x \in X$ we have:*

$$\frac{1}{t} S_t(f)(x) = M_t(f)\left(g_{\ln t}(x)\right).$$

Proof. In order to prove that

$$\frac{1}{t} S_t(f)(x) = M_t(f)(g_{\ln t}(x)), \tag{5.1}$$

for all $t > 0$ and $x \in X$, we first make a change of variables $\boxed{s \leftrightarrow \tilde{s}t}$ to obtain:

$$\begin{aligned} \frac{1}{t} S_t(f)(x) &= \frac{1}{t} \int_0^t f(h_s(x))\, ds \\ &= \frac{1}{t} \int_0^1 f(h_{\tilde{s}t}(x))\, t \cdot d\tilde{s} \\ &= \int_0^1 f(h_{\tilde{s}t}(x))\, d\tilde{s}, \end{aligned}$$

and returning to our 'dummy' variable s, we can write $\frac{1}{t} S_t(f)(x) = \int_0^1 f(h_{st}(x))\, ds$. Using the fact that $g_{-\ln t} \circ h_s = h_{se^{\ln t}} \circ g_{-\ln t} = h_{st} \circ g_{-\ln t}$, we get

$$\int_0^1 f(h_{st}(x))\, ds = \int_0^1 f\left(\left(g_{-\ln t} \circ h_s \circ g_{\ln t}\right)(x)\right)\, ds = M_t(f)(g_{\ln t}(x)),$$

proving equality (5.1) and hence Lemma (24).

□

Now we prove the first statement of Lemma 18, which is, in some extension, the heart of the proof of Theorem 19. It is interesting to observe that, in order to prove equicontinuity of $\mathcal{M} = \{M_t(f)(x)\}_{t \geq 0}$ we make use of the hyperbolicity of the geodesic flow g_t, as we demand points in the same weak unstable leaf of size $\varepsilon > 0$ of g_t remains at most $\varepsilon-$close when iterated by g_{-t}, for some $t \geq 0$.

Lemma. *The family $\mathcal{M} = \{M_t(f)(x)\}_{t \geq 0}$ is equicontinuous.*

Proof. Let $\varepsilon > 0$ and $x_0 \in M$ be fixed arbitrarily and set

$$\omega_f(\varepsilon) = \sup\left\{|f(x) - f(y)| \mid x, y \in M \text{ with } d(x,y) < \varepsilon\right\}.$$

Recall the definition of local (weak) unstable manifold of size $\varepsilon > 0$:

$$W^u_\varepsilon(x) = \{y \in M \mid d(g_{-t}(x), g_{-t}(y)) < \varepsilon, \text{ for all } t \geq 0\},$$

and define

$$K_x = W^u_\varepsilon(x) \cap h_{(-2,3)}\left(B\left(y(x), \delta\right) \cap W^u_\varepsilon(x_0)\right).$$

Now, in the local coordinate system $N(x)$ associated to x, V_x can be written as $V_x = K_x \times (0,1)$ and, from Lemma 20, the measure $\frac{\mu}{\mu(V_x)}$ has a local product structure.

Claim. $|M_t(f)(x) - M_t(f)(x_0)| \leq 2\omega_f(\varepsilon) + \frac{1}{2}\|f\|_2 \cdot \left\| \frac{1}{\mu(V_{x_n})}\chi_{V_{x_n}} - \frac{1}{\mu(V_{x_0})}\chi_{V_{x_0}} \right\|_2.$

If we prove the claim, we are done since $\omega_f(\varepsilon)$ and $\left\|\frac{1}{\mu(V_x)}\chi_{V_x} - \frac{1}{\mu(V_{x_0})}\chi_{V_{x_0}}\right\|_2$ go to 0 as $x \to x_0$, both independent of $t > 0$, from the continuity of f and from Lemma 5.1.

To prove that claim we first notice that $\left|M_t(f)(x) - \int_0^1 \int_{K_x} f(g_{-\ln t}(y,s))\, d\nu_s(y) ds\right| < \omega_f(\varepsilon)$. Indeed,

$$\begin{aligned}
\left|M_t(f)(x) - \int_0^1 \int_{K_x} f(g_{-\ln t}(y,s))\, d\nu_s(y) ds\right| &= \\
\left|\int_0^1 f(g_{-\ln t}(0,s))\, ds - \int_0^1 \int_{K_x} f(g_{-\ln t}(y,s))\, d\nu_s(y) ds\right| &= \\
\left|\int_0^1 \int_{K_x} f(g_{-\ln t}(0,s))\, d\nu_s(y) ds - \int_0^1 \int_{K_x} f(g_{-\ln t}(y,s))\, d\nu_s(y) ds\right| &\leq \\
\int_0^1 \int_{K_x} |f(g_{-\ln t}(0,s)) - f(g_{-\ln t}(y,s))|\, d\nu_s(y) ds &\leq \\
\omega_f(\varepsilon)&,
\end{aligned}$$

since both $(0,s)$ and (y,s) are on the same weak unstable local leaf of $W^u_\varepsilon(h_s(x))$ and, since $W^u_\varepsilon(h_s(x))$ is invariant[1] by g_{-t} for $t \geq 0$, the image of points in $W^u_\varepsilon(h_s(x))$ are still at most $\varepsilon-$close from each other. Hence, we deduce that $|f(g_{-\ln t}(0,s)) - f(g_{-\ln t}(y,s))| \leq \omega_f(\varepsilon)$.

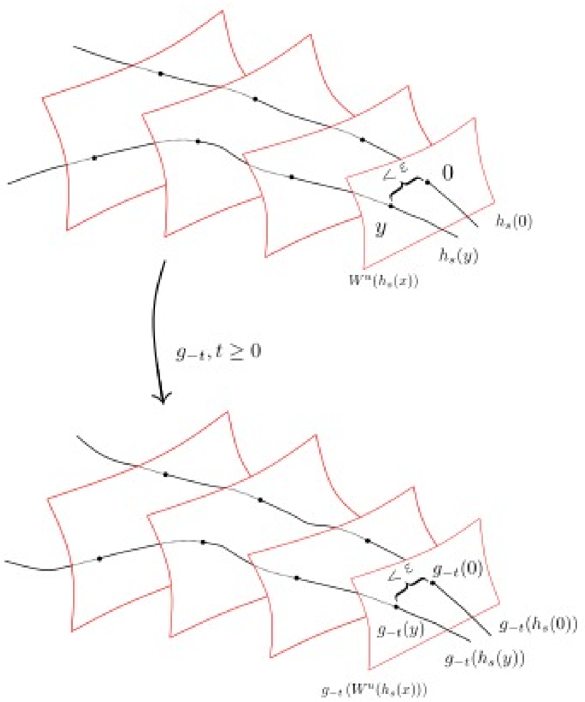

Figure 5.6: Applying g_{-t} for $t \geq 0$ to $W^u_\varepsilon(h_s(x))$.

Adding and subtracting $\int_0^1 \int_{K_x} f(g_{-\ln t}(y,s))\, d\nu_s(y) ds$ and $\int_0^1 \int_{K_{x_0}} f(g_{-\ln t}(y,s))\, d\nu_s(y) ds$ inside

[1]For all $x \in M$, $g_{-t}\left(W^u_\varepsilon(x)\right) \subseteq W^u_\varepsilon\left(g_{-t}(x)\right)$ for all $t \geq 0$.

the modulus $|M_t(f)(x)-M_t(f)(x_0)|$, we get:

$$\begin{aligned}|M_t(f)(x)-M_t(f)(x_0)| = \Big| M_t(f)(x) - \int_0^1\int_{K_x} f(g_{-\ln t}(y,s))\,d\nu_s(y)ds \\ + \int_0^1\int_{K_x} f(g_{-\ln t}(y,s))\,d\nu_s(y)ds - \int_0^1\int_{K_{x_0}} f(g_{-\ln t}(y,s))\,d\nu_s(y)ds \\ + \int_0^1\int_{K_{x_0}} f(g_{-\ln t}(y,s))\,d\nu_s(y)ds - M_t(f)(x_0)\Big|\end{aligned}$$

Using that $\left|M_t(f)(x) - \int_0^1\int_{K_x} f(g_{-\ln t}(y,s))\,d\nu_s(y)ds\right| < \omega_f(\varepsilon)$ and applying the triangular inequality, we get:

$$\begin{aligned}|M_t(f)(x)-M_t(f)(x_0)| \le 2\omega_f(\varepsilon) + \\ \left|\int_0^1\int_{K_x} f(g_{-\ln t}(y,s))\,d\nu_s(y)ds - \int_0^1\int_{K_{x_0}} f(g_{-\ln t}(y,s))\,d\nu_s(y)ds\right|.\end{aligned}$$

Next, we use local product structure of the measure to write:

$$\begin{aligned}\left|\int_0^1\int_{K_x} f(g_{-\ln t}(y,s))\,d\nu_s(y)ds - \int_0^1\int_{K_{x_0}} f(g_{-\ln t}(y,s))\,d\nu_s(y)ds\right| = \\ \left|\frac{1}{\mu(V_x)}\int_{V_x} f\circ g_{-\ln t}\,d\mu - \frac{1}{\mu(V_{x_0})}\int_{V_{x_0}} f\circ g_{-\ln t}\,d\mu\right|.\end{aligned}$$

Then, we have:

$$\begin{aligned}|M_t(f)(x)-M_t(f)(x_0)| &\le 2\omega_f(\varepsilon) + \left|\frac{1}{\mu(V_x)}\int_{V_x} f\circ g_{-\ln t}\,d\mu - \frac{1}{\mu(V_{x_0})}\int_{V_{x_0}} f\circ g_{-\ln t}\,d\mu\right| \\ &= 2\omega_f(\varepsilon) + \left|\frac{1}{\mu(V_x)}\int_X (f\circ g_{-\ln t})\cdot\chi_{V_x}\,d\mu - \frac{1}{\mu(V_{x_0})}\int_X (f\circ g_{-\ln t})\cdot\chi_{V_{x_0}}\,d\mu\right| \\ &\le 2\omega_f(\varepsilon) + \left|\int_X (f\circ g_{-\ln t})\cdot\frac{\chi_{V_x}}{\mu(V_x)}\,d\mu - \int_X (f\circ g_{-\ln t})\cdot\frac{\chi_{V_{x_0}}}{\mu(V_{x_0})}\,d\mu\right| \\ &\le 2\omega_f(\varepsilon) + \left|\int_X (f\circ g_{-\ln t})\cdot\left[\frac{\chi_{V_x}}{\mu(V_x)} - \frac{\chi_{V_{x_0}}}{\mu(V_{x_0})}\right]\,d\mu\right| \\ &\le 2\omega_f(\varepsilon) + \int_X \left|(f\circ g_{-\ln t})\cdot\left[\frac{\chi_{V_x}}{\mu(V_x)} - \frac{\chi_{V_{x_0}}}{\mu(V_{x_0})}\right]\right|\,d\mu.\end{aligned}$$

Now, applying Hölder's inequality with $p = q = 2$, for the $L^2(\mu)-$functions $f\circ g_{-\ln t}$ and $\left[\frac{1}{\mu(V_{x_n})}\chi_{V_{x_n}} - \frac{1}{\mu(V_{x_0})}\chi_{V_{x_0}}\right]$, we have:

$$\int_X \left|(f\circ g_{-\ln t})\cdot\left[\frac{\chi_{V_x}}{\mu(V_x)} - \frac{\chi_{V_{x_0}}}{\mu(V_{x_0})}\right]\right|\,d\mu \le \|f\circ g_{-\ln t}\|_2\cdot\left\|\frac{1}{\mu(V_{x_n})}\chi_{V_{x_n}} - \frac{1}{\mu(V_{x_0})}\chi_{V_{x_0}}\right\|_2.$$

This implies:

$$\begin{aligned}|M_t(f)(x)-M_t(f)(x_0)| &\le 2\omega_f(\varepsilon) + \int_X \left|(f\circ g_{-\ln t})\cdot\left[\frac{\chi_{V_x}}{\mu(V_x)} - \frac{\chi_{V_{x_0}}}{\mu(V_{x_0})}\right]\right|\,d\mu \\ &\le 2\omega_f(\varepsilon) + \|f\circ g_{-\ln t}\|_2\cdot\left\|\frac{1}{\mu(V_{x_n})}\chi_{V_{x_n}} - \frac{1}{\mu(V_{x_0})}\chi_{V_{x_0}}\right\|_2.\end{aligned}$$

Since μ is g_t-invariant, we must have:

$$\|f \circ g_{-\ln t}\|_2 = \|f\|_2.$$

Hence,

$$\begin{aligned} |M_t(f)(x) - M_t(f)(x_0)| &\leq 2\omega_f(\varepsilon) + \|f \circ g_{-\ln t}\|_2 \cdot \left\| \frac{1}{\mu(V_{x_n})}\chi_{V_{x_n}} - \frac{1}{\mu(V_{x_0})}\chi_{V_{x_0}} \right\|_2 \\ &\leq 2\omega_f(\varepsilon) + \|f\|_2 \cdot \left\| \frac{1}{\mu(V_{x_n})}\chi_{V_{x_n}} - \frac{1}{\mu(V_{x_0})}\chi_{V_{x_0}} \right\|_2. \end{aligned}$$

This proves the claim and also the Lemma. □

Lemma. *Let $\overline{f}$ be an accumulation point for the family $\mathcal{M} = \{M_t(f)(x)\}_{t\geq 0}$ on $C^0(X)$ and suppose h_s has a dense orbit. Then $\overline{f}$ must be constant.*

Proof. Let $\overline{f}$ be an accumulation point for $\mathcal{M}$ and $(t_n)_n$ be a sequence such that $t_n \to +\infty$. Then, $\|M_{t_n}(f) - \overline{f}\|_\infty \to 0$ and, in particular,

$$\|M_{t_n}(f) - \overline{f}\|_2 \to 0$$

as $n \to +\infty$. Using equality $\frac{1}{t}S_t(f)(x) = M_t(f)(g_{\ln t}(x))$ from Lemma 24 and using the fact that μ is $g-$invariant, we have:

$$\left\| \frac{1}{t_n}S_{t_n}(f) - \overline{f} \circ g_{\ln t_n} \right\|_2 = \left\| M_{t_n}(f) \circ g_{\ln t_n} - \overline{f} \circ g_{\ln t_n} \right\|_2 = \|M_{t_n}(f) - \overline{f}\|_2 \to 0,$$

as $n \to +\infty$.

Now, recalling that $S_t(f)$ is the Birkhoff sum of the horocycle flow h_s and applying von Neumann Ergodic Theorem (Theorem 3), we conclude that there exists an h_s-invariant function $P_h(f) \in L^2(\mu)$ such that $\left\|\frac{1}{t}S_t(f) - P_h(f)\right\|_2 \to 0$ as $t \to +\infty$. In particular, we conclude that

$$\left\|\overline{f} \circ g_{\ln t_n} - P_h(f)\right\|_2 \to 0$$

when $n \to +\infty$. Once again using the $g-$invariance of the measure μ we have

$$\left\|\overline{f} - P_h(f) \circ g_{-\ln t_n}\right\|_2 = \left\|\overline{f} \circ g_{\ln t_n} - P_h(f)\right\|_2 \xrightarrow[n\to+\infty]{} 0.$$

Since g_t sends h_s-orbits into h_s-orbits and since $P_h(f)$ is h_s-invariant, for each $t \in \mathbb{R}$ the composition $P_h(f) \circ g_t$ is still a h_s-invariant function. Moreover, applying Lemma 3 to this setting, we conclude that, since $\overline{f}$ is the L^2-limit of h_s-invariant functions, $\overline{f}$ will itself be h_s-invariant. Since there exists a dense h_s-orbit and since $\overline{f} \in C^0(X)$, then $\overline{f}$ is constant. This concludes the proof of this Lemma and also the proof of Lemma 18. □

Lemma 19

Lemma. *With respect to $S_t(f)$ we have:*

(1) *the family $\mathcal{S} = \left\{\frac{1}{t}S_t(f)(x)\right\}_t$ has compact closure on $C^0(X)$;*

(2) *each accumulation point of $\mathcal{S}$ is constant.*

Proof. To check both items (1) and (2), take a sequence $(t_n)_n$ with $t_n \to +\infty$. From Lemma 18 there are a subsequence $(t_{n_k})_k$ and a constant $c \in \mathbb{R}$ such that

$$M_{t_{n_k}}(f) \xrightarrow{unif} c.$$

So, given $\varepsilon > 0$ there is an order $k_0 \in \mathbb{N}$ such that, if $k \geq k_0$, then: $\left|M_{t_{n_k}}(f)(x) - c\right| < \varepsilon$, for all $x \in M$. Hence, for $x \in M$, we have

$$\left|\frac{1}{t_{n_k}} S_{t_{n_k}}(f)(x) - c\right| = \left|M_{t_{n_k}}(f)(g_{\ln t_{n_k}}(x)) - c\right| < \varepsilon,$$

for all $k \geq k_0$.

□

5.2 Horocycle flows are uniquely ergodic

Now we apply Theorem 19 to the context of horocycle flow associated to a geodesic flow on the unit tangent bundle of a surface with constant negative curvature. Even though constant negative curvature implies the geodesic flow to be an Anosov flow, at first glace we cannot say much about the dynamics of the horocycle flows associated to this geodesic flow.

In Chapter 4 we obtained a first property about its dynamics: they are minimal (i.e., every orbit is dense). In this Chapter 5 we obtained another information, from the point of view of ergodic theory: they admit an unique invariant probability measure.

To be more precise, in Chapter 3 we have shown that the Liouville measure on a compact negatively curved manifold M is invariant under both the geodesic g_t and the horocycle h_s flows. As we claimed, Theorem 19 guarantees that, in this context, it is the unique ergodic measure invariant under h_s:

Theorem 20. *The horocycle flow associated to a geodesic flow $g_t \colon T^1S \to T^1S$ on a surface with constant negative curvature is uniquely ergodic.*

Proof. Since the geodesic flow $g_t \colon T^1S \to T^1S$ on a surface with constant negative curvature is an Anosov flow with dimension one unstable foliation. From Proposition 18, the horocycle flow satisfy the relation

$$g_t \circ h_s = h_{se^{-t}} \circ g_t,$$

for every $t, s \in \mathbb{R}$. Finally, by Theorem 18, h_s has a dense orbit. Hence, we are in the hypotheses of Theorem 19: this shows that h_s is uniquely ergodic. □

APPENDIX **A**

Anosov diffeomorphisms: transitivity, minimality and topological mixing

A.1 Anosov diffeomorphisms

Let M be a closed Riemannian manifold. A diffeomorphism $f\colon M \to M$ is called Anosov if there exists a $f-$invariant hyperbolic splitting

$$TM = E^u \oplus E^s,$$

i.e., each bundle E^u and E^s, is preserved by df, and the vectors on E^u are exponentially expanded by df and the vectors on E^s are exponentially contracted by df. In other words, there exists constants $C \geq 0$ and $0 < \lambda < 1$, such that:

$$\|df^{-n}(v)\| \leq C\lambda^n \|v\|, \text{ for all } v \in E^u \text{ and } n \geq 0,$$

and

$$\|df^{n}(v)\| \leq C\lambda^n \|v\|, \text{ for all } v \in E^s \text{ and } n \geq 0.$$

As in the case of flows, the spaces E^u and E^s are called unstable and stable spaces, respectively.

A well-known example of Anosov diffeomorphism is the Anosov Cat Map, that we presented in Chapter 3 and that we here investigate with more details, following [Wen16]. To do so, first recall that an invertible matrix $A\colon \mathbb{R}^n \to \mathbb{R}^n$ is called ***hyperbolic*** if it has no eigenvalue of absolute value 1.

Definition 24 (Anosov automorphism)**.** *A linear map* $A\colon \mathbb{R}^2 \to \mathbb{R}^2$ *is called an Anosov automorphism if* $\det A = \pm 1$*, A has integer entries and is hyperbolic.*

Proposition 20. *If* $A\colon \mathbb{R}^2 \to \mathbb{R}^2$ *is an Anosov automorphism, then the eigenvalues of A are irrational numbers* $\lambda_s, \lambda_u \in \mathbb{R}$ *such that* $|\lambda_s| < 1 < |\lambda_u|$*, and the slopes of the two eigenspaces are irrational.*

Proof. There are three options for the eigenvalues λ_s and λ_u: or they are complex conjugate, or they are the same, or they are real and distinct. We want to show that they satisfy the last condition.

Since $\det A = \lambda_s \cdot \lambda_s$ and $|\det A| = 1$, if we were in the first two cases, this would imply $|\lambda_s| = 1 = |\lambda_u|$, contradicting the hypothesis that A is hyperbolic. This proves the eigenvalues λ_u and λ_s are real and distinct and we can suppose, without loss of generality, that $|\lambda_s| < 1 < |\lambda_u|$.

To see that λ_s is irrational (the proof for λ_u is analogous), suppose $\lambda_s = \frac{p}{q}$, where $p, q \in \mathbb{Z}$, $q \neq 0$ and $\gcd(p, q) = 1$. Note that, since $|\lambda_s \cdot \lambda_u| = 1$, we must also have $p \neq 0$. Now, observe that

$$\begin{aligned} 0 &= \det\left(A - \frac{p}{q} I\right) \\ &= \frac{p^2}{q^2} - \frac{p}{q} \cdot \operatorname{tr} A + \det A \\ &= \frac{p^2}{q^2} - \frac{p}{q} \cdot \operatorname{tr} A \pm 1, \end{aligned}$$

so $0 = p^2 - pq \operatorname{tr} A \pm q^2$. This equality implies:

$$q(p \operatorname{tr} A \mp q) = p^2,$$

and

$$p(-p + q \operatorname{tr} A) = \pm q^2,$$

which imply, since $\gcd(p, q) = 1$, $q|p$ and $p|q$. But this would imply $|\lambda_s| = 1$, contradicting once again the hyperbolicity of A. Since an analogous proof works for λ_u, we conclude that λ_s and λ_u are irrational.

Finally, for the statement on the slope of the eigenvalues, let $v = (v_1, v_2)$ be an eigenvector of λ_s. First note that v indeed has a slope, i.e., $v_1 \neq 0$. To see that, suppose the matrix A is of the form:

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}.$$

If $v_1 = 0$ then $Av = \lambda_1 v$ implies $a_{22} = \lambda_s$, contradicting the hypothesis that A has only integer entries. So, $v_1 \neq 0$ and we can suppose $v = (1, \alpha)$.

To prove the slope of the eigenspace associated to λ_s is irrational, we are going to show that $\alpha \in \mathbb{R} \setminus \mathbb{Q}$. Since, once again, $Av = \lambda_s v$, we get:

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} 1 \\ \alpha \end{pmatrix} = \begin{pmatrix} \lambda_s \\ \lambda_s \cdot \alpha \end{pmatrix}.$$

Thus, $a_{11} + a_{12} \cdot \alpha = \lambda_s$ and, if α were rational, we would have λ_s also rational. Hence, $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ as we claimed. The proof for the slope of the eigenspace of λ_u is similar.

□

Note that, since it has only integer entries, an Anosov automorphism $A\colon \mathbb{R}^2 \to \mathbb{R}^2$ preserves the lattice $\mathbb{Z}^2$, i.e., $A(\mathbb{Z}^2) \subset \mathbb{Z}^2$. In particular, for any vectors $v \in \mathbb{R}^2$ and $n \in \mathbb{Z}^2$, we have:

$$A(v + n) - A(v) = A(n) \in \mathbb{Z}^2.$$

This allow us to define a map $f_A\colon \mathbb{T}^2 \to \mathbb{T}^2$ on the torus that makes the diagram

$$\begin{tikzcd} \mathbb{R}^2 \arrow[r, "A"] \arrow[d, "\pi"'] & \mathbb{R}^2 \arrow[d, "\pi"] \\ \mathbb{T}^2 \arrow[r, "f_A"'] & \mathbb{T}^2 \end{tikzcd}$$

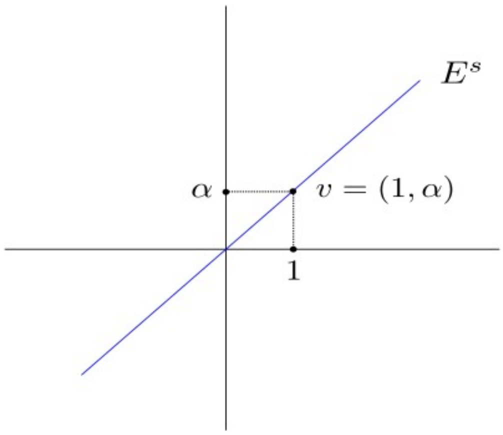

Figure A.1: The space E^s.

to commute, i.e., $\pi \circ A = f_A \circ \pi$. Here $\pi \colon \mathbb{R}^2 \to \mathbb{T}^2$ is the canonical projection to the quotient, that is, the map that to each $v \in \mathbb{R}^2$ associates its equivalence class on the quotient $\mathbb{R}^2/\mathbb{Z}^2$:

$$\pi(v) = \{w \in \mathbb{R}^2 \mid w - v \in \mathbb{Z}^2\} = [v].$$

The map f_A is a C^∞ diffeomorphism. Indeed, it is a C^∞ map and, if we suppose A is once again of the form

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix},$$

the A^{-1} will be:

$$A^{-1} = \frac{1}{\det A} \cdot \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}.$$

Since $|\det A| = 1$, all the entries of A^{-1} are still integers and A^{-1} also induces a C^∞ map on $\mathbb{T}^2$ which is the inverse of f_A, i.e.,

$$(f_A)^{-1} = f_{A^{-1}}.$$

Example 14 (The Cat Map)**.** *Consider the Anosov automorphism $A \colon \mathbb{R}^2 \to \mathbb{R}^2$ defined by*

$$Av = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \cdot v.$$

The induced map f_A on $\mathbb{T}^2$ is called Anosov's Cat Map.

Example 15. *Another example of Anosov diffeomorphism on the torus is the map f_B induced be the Anosov automorphism $B \colon \mathbb{R}^2 \to \mathbb{R}^2$ defined by*

$$Bv = \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix} \cdot v.$$

This follows from the fact that A has integer entries, $\det B = 4 - 3 = 1$ and its eigenvalues are $\lambda_2 = 2 - \sqrt{3}$ and $\lambda_1 = 2 + \sqrt{3}$, so that

$$0 < \lambda_2 < 1 < \lambda_1.$$

Whenever we have an Anosov automorphism $A\colon \mathbb{R}^2 \to \mathbb{R}^2$, the projection to the torus $\mathbb{T}^2$ generates an Anosov diffeomorphism:

Proposition 21. *An Anosov toral automorphism $f_A\colon \mathbb{T}^2 \to \mathbb{T}^2$ induced by an Anosov automorphism $A\colon \mathbb{R}^2 \to \mathbb{R}^2$ is an Anosov diffeomorphism.*

Proof. Since $(df_A)_x = A$ for every point x in the torus, we have a f_A-invariant hyperbolic decomposition $T_x\mathbb{T}^2 = E^s \oplus E^u$ originated from A and the previous Proposition 20 guarantees this decomposition contract vectors on E^s and expands vectors on E^u. $\square$

Now, we present the notion of stable and unstable spaces for toral automorphisms. A brief study of these sets will allow us to obtain further information about the dynamics of f_A.

Given a hyperbolic matrix $A\colon \mathbb{R}^2 \to \mathbb{R}^2$ and a vector $v \in \mathbb{R}^2$, define

$$W^s(v, A) = \{w \in \mathbb{R}^2 \mid \lim_{n\to+\infty} \|A^n w - A^n v\| \to 0\}$$

and

$$W^u(v, A) = \{w \in \mathbb{R}^2 \mid \lim_{n\to+\infty} \|A^{-n} w - A^{-n} v\| \to 0\},$$

the *stable* and *unstable spaces* of v with respect to A.

Similarly, given a point $x \in \mathbb{T}^2$, define the (global) *stable* and *unstable manifolds* of x with respect to f_A as the sets:

$$W^s(x, f_A) = \{y \in \mathbb{T}^2 \mid \lim_{n\to+\infty} d(f_A^n(x), f_A^n(y)) \to 0\}$$

and

$$W^u(x, f_A) = \{y \in \mathbb{T}^2 \mid \lim_{n\to+\infty} d(f_A^{-n}(x), f_A^{-n}(y)) \to 0\},$$

respectively. These set are $f-$invariant, in the sense that $f_A\left(W^s(x, f_A)\right) = W^s(f_A(x), f_A)$ and $f_A\left(W^u(x, f_A)\right) = W^u(f_A(x), f_A)$.

The next theorem relates these two notions (for A and f_A) and gives more details on the geometry of the stable and unstable manifolds of a point in $\mathbb{T}^2$ with respect an Anosov toral automorphism.

Theorem 21. *Let $A\colon \mathbb{R}^2 \to \mathbb{R}^2$ be an Anosov automorphism and $f_A\colon \mathbb{T}^2 \to \mathbb{T}^2$ be the induced toral automorphism. Then:*

(i) *for any $v \in \mathbb{R}^2$, $W^s(v, A) = v + E^s$ and $W^u(v, A) = v + E^u$, where $\mathbb{R}^2 = E^s \oplus E^u$ is the hyperbolic splitting of A.*

(ii) *for any $x \in \mathbb{T}^2$, $W^s(x, f_A) = \pi(W^s(v, A))$, where v is an arbitrary point $v \in \pi^{-1}(x)$. The same holds for the unstable manifold.*

(iii) *$W^s(x, f_A)$ is an immersed C^∞ submanifold that is dense in $\mathbb{T}^2$. Likewise for $W^u(x, f_A)$. Moreover, for any $x, y \in \mathbb{T}^2$, $W^s(x, f_A)$ intersects $W^u(y, f_A)$ transversely at a dense subset of $\mathbb{T}^2$.*

Proof. To prove the assertion in (i), take $w \in W^s(v, A)$. Then, $\|A^n(w - v)\| = \|A^n w - A^n v\| \to 0$ as $n \to +\infty$. So, $w - v \in E^s$, i.e., $w \in v + E^s$. This proves $W^u(v, A) \subseteq v + E^u$. Reciprocally, let $w \in v + E^u$. Hence, $\|A^n w - A^n v\| = \|A^n(w - v)\| \to 0$ as $n \to +\infty$, proving that $w \in W^s(v, A)$ and that $W^s(v, A) = v + E^s$. The case for the unstable set is similar.

Now, fix $v \in \mathbb{R}^2$ such that $x = \pi(v)$, fix $v \in \pi^{-1}(x)$. In order to prove $W^s(x, f_A) = \pi(W^s(v, A))$ we start by proving $W^s(x, f_A) \supseteq \pi(W^s(v, A))$.

Let $w \in W^s(v, A)$. Then, $\|A^n w - A^n v\| \to 0$ as $n \to +\infty$. Since the projection π is uniformly continuous, we have $d(\pi(A^n w), \pi(A^n v)) \to 0$ as $n \to +\infty$ and then

$$d(f_A^n(\pi(w)), f_A^n(x)) \to 0,$$

as $n \to +\infty$. This proves that $\pi(w) \in W^s(x, f_A)$ and, since w was taken arbitrarily, that $\pi(W^s(v, A)) \subseteq W^s(x, f_A)$.

We now prove that $W^s(x, f_A) \subseteq \pi(W^s(v, A))$. To do so, note that if we take $\varepsilon = \frac{1}{2} > 0$ then for any $v, w \in \mathbb{R}^2$ such that $\|w - v\| \leq \varepsilon$ then $d(\pi(w), \pi(v)) = \|w - v\|$. Now, take $0 < \delta < \varepsilon$ such that for any $v, w \in \mathbb{R}^2$ with $\|w - v\| \leq \delta$ then $\|Aw - Av\| \leq \varepsilon$. Such δ can be taken by the uniform continuity of A.

Now let $y \in W^s(x, f_A)$. We are going to find $w \in W^s(v, A)$ such that $\pi(w) = y$. Since $y \in W^s(x, f_A)$, $d(f_A^n(y), f_A^n(x)) \to 0$ as $n \to +\infty$. Take $m \in \mathbb{N}$ such that

$$d(f_A^n(y), f_A^n(x)) \leq \delta,$$

for all $n \geq m$.

Since $\pi(A^m v) = f^m(x)$, there is a unique $z \in B(A^m v, \varepsilon)$ such that $\pi(z) = f^m(y)$. By taking $w = A^{-m}(z)$, we have:

$$\pi(w) = \pi(A^{-m} z) = f^{-m}(\pi(z)) = f^{-m}(f^m(y)) = y.$$

So, if we prove that $w \in W^s(v, A)$ we are done. This is equivalent to prove that $z \in W^s(A^m v, A)$, and that is what we shall do. Since $\|z - A^m v\| \leq \varepsilon$, we have

$$\|z - A^m v\| = d(\pi(z), \pi(A^m(v))) = d(f^m(y), f^m(x)) \leq \delta.$$

By the choice of δ, we get $\|Az - A(A^m v)\| \leq \varepsilon$. But then,

$$\|Az - A(A^m v)\| = d(\pi(Az), \pi(A(A^m v))) = d(f^{m+1}(y), f^{m+1}(x)) \leq \delta.$$

Proceeding inductively, we finally get:

$$\|A^n z - A^n(A^m v)\| = d(f^{m+n}(y), f^{m+n}(x)) \leq \delta.$$

Since, $d(f^{m+n}(y), f^{m+n}(x)) \to 0$ as $n \to +\infty$, we conclude that $z \in W^s(A^m v, A)$, or equivalently, that $w \in W^s(v, A)$, concluding the proof that $W^s(x, f_A) \subseteq \pi(W^s(v, A))$, and hence that $W^s(x, f_A) = \pi(W^s(v, A))$. Once again, the unstable case is analogous. This proves item (ii) of the theorem.

Finally, we show (iii). The set $W^s(v, A)$ is a line in $\mathbb{R}^2$. Since the projection $\pi \colon \mathbb{R}^2 \to \mathbb{T}^2$ is a C^∞ embedding, the set $W^s(x, f_A)$ is an immersed C^∞ submanifold of $\mathbb{T}^2$.

For the denseness part, we use a fact we proved in Chapter 2. There we presented Proposition 4, which states the following:

Proposition. *A linear flow $\varphi_t(x) = [x + t\theta]$ on $\mathbb{T}^n$ is minimal if, and only if, the components of θ are rationally independent.*

Recall that the components of the vector $\theta = (\theta_1, \dots, \theta_n)$ are rationally independent if $k \in \mathbb{Z}^n$ and $\langle k, v \rangle = 0$, then $k = 0$.

Back to the setting of f_A suppose, without loss of generality, that $x = \pi(0)$. We claim the set $W^s(x, f_A)$ coincides with the orbit of a minimal flow on $\mathbb{T}^2$ and, therefore, is a dense set on $\mathbb{T}^2$.[1]

Claim. *The set $W^s(x, f_A)$ coincides with a orbit of a linear flow $\varphi_t(x) = [x + t\theta]$ on $\mathbb{T}^2$, for θ with rationally independent components.*

Proof of the Claim. In this case, where we suppose $x = \pi(0)$, we have $W^s(0, A) = E^s$ and, by Proposition 20 above, we know that E^s is a line through the origin of $\mathbb{R}^2$ generated by a vector $(1, \alpha)$, with $\alpha \in \mathbb{R} \setminus \mathbb{Q}$. Then,

$$E^s = \{t \cdot \Theta \mid t \in \mathbb{R}\},$$

where $\Theta := (1, \alpha) \in \mathbb{R}^2$ has rationally independent components.

Since $W^s(x, f_A) = \pi\left(W^s(0, A)\right) = \pi(E^s)$, we have:

$$\begin{aligned} W^s(x, f_A) &= \{\pi(0 + t \cdot \Theta) \mid t \in \mathbb{R}\} \\ &= \{\pi(0) + t \cdot \pi(\Theta) \mid t \in \mathbb{R}\} \\ &= \{[x + t \cdot \theta] \mid t \in \mathbb{R}\}, \end{aligned}$$

where $\theta = \pi(\Theta)$. Observe that since the components of Θ are rationally independent, the same holds for θ. This proves the claim. □

Lastly, the transversality condition follow from the fact that for any $w, v \in \mathbb{R}^2$, the lines $W^s(v, A)$ and $W^u(w, A)$ intersect transversely. Since π is a local embedding, $W^s(x, f_A)$ and $W^u(y, f_A)$ intersect (in a dense set) transversely. This concludes the proof of item (iii) and, therefore, the proof of the theorem. □

A fact we extract from Theorem 21 is a result we will proof in a more general setting in Section A.3. An toral automorphism $f_A \colon \mathbb{T}^2 \to \mathbb{T}^2$ has the property of being **minimal**, i.e., all stable and unstable manifolds are dense on $\mathbb{T}^2$. The main goal of Section A.3 is to prove an analogous for Anosov diffeomorphisms. We use the next Section to introduce several properties that will be important for this proof.

A.2 Some properties of Anosov diffeomorphisms

In this section we describe some properties that all Anosov diffeomorphisms share. Inspired on what we have done for the Anosov toral automorphisms in Theorem 21, we provide a brief discussion aimed to understand a little more of the structure of the stable and unstable manifolds of an Anosov diffeomorphism. Once again, we restrict our context to a diffeomorphism $f \colon M \to M$ defined on a Riemannian closed manifold.

Definition 25. *Given a point $x \in M$, its $\omega-$**limit** is the set of all $y \in M$ such that there is an infinite sequence $(n_i)_{i \in \mathbb{N}}$, with $n_i \to \infty$, such that $f^{n_i}(x) \to y$. We denote it by $\omega(x)$.*

[1] Observe that this is the same sort of idea used to prove Theorem 18: there we showed that the orbit of the horocycle flow coincides with a dense set. Here we are showing that a set coincides with a dense orbit of a flow.

Definition 26. *A compact invariant*[2] *subset $\Lambda \subseteq M$ is called **transitive** if there is some point $x \in \Lambda$ such that $\omega(x) = \Lambda$.*

An equivalent definition of transitivity for diffeomorphisms is the following:

Proposition 22. *A compact invariant subset $\Lambda \subset M$ is transitive if and only if for any two open sets $U, V \subseteq \Lambda$ exists $n \geq 1$ such that $f^n(U) \cap V \neq \emptyset$.*

Definition 27. *A compact invariant subset $\Lambda \subseteq M$ is called **topologically mixing** if for any two relative open sets U and V of Λ, there is an integer $N = N(U, V) \geq 1$ such that $f^n(U) \cap V \neq \emptyset$ for all $n \geq N$.*

Definition 28 (Local Stable Manifold)**.** *For a point $x \in M$ and $r > 0$, define the **local stable manifold** of x of size r with respect to f to be*

$$W_r^s(x) = \left\{ y \in M \mid d(f^n(y), f^n(x)) \leq r \text{ for all } n \geq 0, \text{ and } \lim_{n \to +\infty} d(f^n(y), f^n(x)) = 0 \right\}.$$

*Similarly, define the **local unstable manifold** of x of size r with respect to f to be*

$$W_r^u(x) = \left\{ y \in M \mid d(f^{-n}(y), f^{-n}(x)) \leq r \text{ for all } n \geq 0, \text{ and } \lim_{n \to +\infty} d(f^{-n}(y), f^{-n}(x)) = 0 \right\}.$$

There will be several occasions where we won't need to specify the size r of the local stable and unstable manifold. When this happens we simply write $W_{loc}^s(x)$ and $W_{loc}^u(x)$, respectively.

For an Anosov diffeomorphism $f\colon M \to M$, we have the following characterization for the local stable and unstable manifolds:

Proposition 23. *There are uniform constants $r > 0$, $C \geq 1$, and $0 < \lambda < 1$ such that for any $x \in \Lambda$:*

$$\begin{aligned} W_r^s(x) &= \{ y \in M \mid d(f^n(y), f^n(x)) \leq r, \text{ for all } n \geq 0 \} \\ &= \{ y \in M \mid d(f^n(y), f^n(x)) \leq r, \text{ and } d(f^n(y), f^n(x)) \leq C\lambda^n d(x, y), \text{ for all } n \geq 0 \}, \end{aligned}$$

and similarly:

$$\begin{aligned} W_r^u(x) &= \{ y \in M \mid d(f^{-n}(y), f^{-n}(x)) \leq r, \text{ for all } n \geq 0 \} \\ &= \left\{ y \in M \mid d(f^{-n}(y), f^{-n}(x)) \leq r, \text{ and } d(f^{-n}(y), f^{-n}(x)) \leq C\lambda^n d(x, y), \text{ for all } n \geq 0 \right\}. \end{aligned}$$

Proof. For a proof see Theorem 4.13, on p. 101 of [Wen16]. □

We now introduce a structure that, for the case of Anosov diffeomorphisms, the local stable and unstable manifolds endow M with:

Proposition 24. *Let $f\colon M \to M$ be an Anosov diffeomorphism. Then M has a product structure, i.e., there are small enough $\varepsilon > 0$ and $\delta > 0$ such that:*

(i) for all x and y on M, the intersection $W_\varepsilon^s(x) \cap W_\varepsilon^u(y)$ consists of at most a point;

(ii) for all x and y on M such that $d(x, y) < \delta$ the intersection above consists of exactly one point, denoted $[x, y] = W_\varepsilon^s(x) \cap W_\varepsilon^u(y)$, and the intersection is transverse.

Proof. For a proof see, for example, Proposition 5.9.3 on p. 129 of [BS02]. □

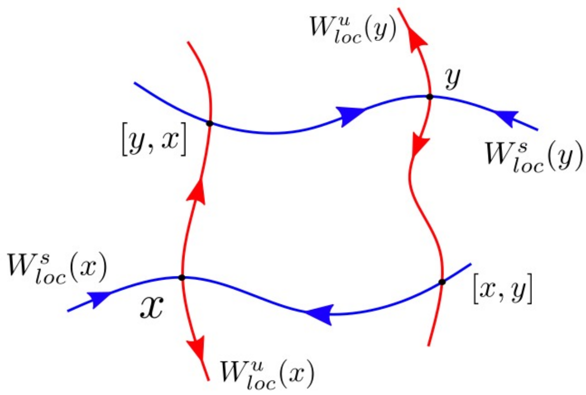

Figure A.2: The local product structure guarantees the existence of the points $[x, y] = W^s_{loc}(x) \cap W^u_{loc}(y)$ and $[y, x] = W^s_{loc}(y) \cap W^u_{loc}(x)$.

Next, we follow once more [Wen16] to state a very important theorem that provides a geometrical structure to a dynamical set.

Theorem 22 (Stable Manifold Theorem)**.** *Let $f\colon M \to M$ be a C^k ($k \geq 1$) Anosov diffeomorphism on M with splitting $TM = E^s \oplus E^u$. Then, there is $r > 0$ such that, for every $x \in M$:*

(i) *$W^s_r(x)$ is a C^k embedded submanifold of M of dimension $\dim E^s(x)$ tangent at x to $E^s(x)$, and $W^s_r(x)$ varies continuously in $x \in M$ with respect to the C^k topology.*

(ii) *the family $\{W^s_r(x)\}_{x\in M}$ is **self-coherent**, i.e., for any $x, y \in M$,*

$$\operatorname{int} W^s_r(x) \cap \operatorname{int} W^s_r(y)$$

is open in both $W^s_r(x)$ and $W^s_r(y)$.

(iii) *the **global stable manifold** $W^s(x)$, i.e., the set*

$$W^s(x) = \left\{ y \in M \mid \lim_{n\to+\infty} d(f^n(y), f^n(x)) \to 0 \right\},$$

is an immersed C^k submanifold of M of dimension $\dim E^s(x)$.

Proof. Proofs of the stable manifold theorem for diffeomorphisms can be found in many books on dynamical systems, such as [BS02], [KH97] and [Wen16]. $\square$

Note that, for all $r > 0$, we have the following relation between the local and global stable and unstable manifold:

$$W^s(x) = \bigcup_{n\geq 0} f^{-n}\left(W^s_r\left(f^n(x)\right)\right),$$
$$W^u(x) = \bigcup_{n\geq 0} f^{n}\left(W^u_r\left(f^{-n}(x)\right)\right).$$

[2] We say that that a subset $\Lambda \subseteq M$ is **invariant** by f is $f(\Lambda) = \Lambda$.

A.3 Minimal Anosov diffeomorphisms

Whenever $f\colon M \to M$ is an Anosov diffeomorphism, we can apply the Stable Manifold Theorem for each point and the self-coherence of the families $\{W^s_r(x)\}$ and $\{W^u_r(x)\}$ will imply that both $W^u(x)$ and $W^s(x)$ give rise to invariant foliations: the unstable foliation $\mathcal{F}^u$, whose leafs are $W^u(x)$ and the stable foliation $\mathcal{F}^s$, whose leafs are $W^s(x)$.

Definition 29. *An Anosov diffeomorphism is called* ***minimal*** *if its global stable and unstable leafs through every point is dense on M.*

> **Warning:** as in the case of flows, we will call a diffeomorphism **minimal** in two different contexts. It will have the meaning that all its orbits are dense, that is the notion for a general dynamical system. Also, it will have the meaning that all stable and unstable leafs are dense, that will be used for Anosov diffeomorphism. Observe that an Anosov diffeomorphism cannot be minimal in the sense that all its orbits are dense, since it has periodic points. From the context, it will be clear to which meaning we are referring to.

In Chapter 4 we have shown that a transitive Anosov flow can either be minimal or be the suspension of a diffeomorphism. This dichotomy cannot occur in the case of Anosov diffeomorphisms.

Theorem 23. *Let $f\colon M \to M$ be a transitive Anosov diffeomorphism on a compact and connected Riemannian manifold M. Then,*

$$\overline{W^u(x)} = \overline{W^s(x)} = M,$$

for all $x \in M$. In other words, if f is transitive Anosov diffeomorphism, then it is minimal.

Proof. The proof will follow from three lemmas, starting by:

Lemma 25. *For all $x \in M$ we have $\overline{W^u\left(\mathcal{O}\left(x\right)\right)} = M$, where $W^u\left(\mathcal{O}\left(x\right)\right) := \bigcup_{y\in\mathcal{O}(x)} W^u(y)$.*

Proof. Fix $x \in M$ and choose arbitrary $y \in M$ and $r > 0$. Since we are supposing f to be transitive, there exists $p \in M$ such that $\omega(p) = M$. In particular, there exists $n \geq 0$ such that $f^n(p)$ belongs to a product neighborhood around x and $m > n$ such that $f^m(p) \in B_r(y)$.

Since $f^n(p)$ belongs to a product neighborhood around x, its local stable neighborhood $W^s_{loc}(f^n(p))$ intersects the local unstable neighborhood $W^u_{loc}(x)$ of x in a unique point $z \in W^s_{loc}(f^n(p)) \cap W^u_{loc}(x)$.

Now, from the fact that $z \in W^s_{loc}(f^n(p))$, we must have $d(f^k(f^n(p)), f^k(z)) < d(f^n(p), z)$ for all $k > 0$. Since $m > n$,

$$d(f^m(p), f^{m-n}(z)) = d(f^{m-n}(f^n(p)), f^{m-n}(z)) < d(f^n(p), z).$$

So, by refining our choice of n in the beginning so that $d(f^n(p), z) < r/2$, we have concluded that $d(y, f^{m-n}(z)) < r$.

On the other hand, $z \in W^u_{loc}(x)$ and hence $f^{m-n}(z) \in W^u\left(\mathcal{O}\left(x\right)\right)$. So we have shown that, for all $y \in M$ and all $r > 0$, there is $q \in W^u(\mathcal{O}\left(x\right)) \cap B_r(y)$.

□

A direct corollary of this lemma is:

Corollary 8. *For all periodic point $p \in M$, $\overline{W^u(\mathcal{O}(p))} = M$.*

Remark 1. *Notice that the statement of the Corollary does not demand the existence of a periodic point to be true. However, that there exists, in fact, a periodic point for f is a simple consequence of the fact that f is transitive and from a very powerful theorem: the Anosov's Closing Lemma.*

Theorem 24 (Anosov Closing Lemma)**.** *The set of periodic points for an Anosov diffeomorphism f is dense on $\Omega(f)$, i.e.,*

$$\overline{\mathbf{Per}(f)} = \Omega(f).$$

Proof. See [Wen16], p. 132, Theorem 4.28. $\square$

The next lemma we need to prove Theorem 23 will be a consequence if another celebrated and useful theorem, with a very geometrical flavor.

In order to present it, we need some notation: for a hyperbolic fixed point $p \in M$ of f, we call $u = \dim W^u(p)$ and a $u-$dimensional C^1 embedded disc in M a $u-$disc. Likewise for $s-$disc and $s = \dim W^s(p)$.

Theorem 25 ($\lambda-$Lemma)**.** *Let $p \in M$ be a hyperbolic fixed point of $f\colon M \to M$. For any $u-$disc B in $W^u(p)$, any point $x \in W^s(p)$, any $u-$disc D transverse to $W^s(p)$ at x, and any $\varepsilon > 0$, there is $N > 0$ such that if $n > N$, $f^n(D)$ contains a $u-$disc that is C^1 $\varepsilon-$close to B.*

So we fix a such periodic point (that we henceforth suppose, without loss of generality, a fixed point for f) and will use the fact that $\overline{W^u(p)} = M$ to show that $\overline{W^u(x)} = M$ for all $x \in M$.

Lemma 26. *There is $m \in \mathbb{N}$ such that for all $x \in M$ the unstable manifold $W^u(f^m(x))$ of $f^m(x)$ intersects the local stable manifold $W^s_{loc}(p)$ of p transversely.*

Remark 2. *It's worth to emphasize that the order $m \in \mathbb{N}$ that we obtain in the Lemma, does **not** depend on the point: the same m works for all $x \in M$. This uniformity is essential to what comes ahead.*

Proof. Since $\overline{W^u(\mathcal{O}(x))} = M$ for all $x \in M$, there exists $m = m(x) \in \mathbb{N}$ such that $W^u(f^m(x)) \cap B_r(p) \neq \emptyset$ for some $r-$ball $B_r(p)$ centered in p inside a product neighborhood around p.

By continuity of f^m, for every x' close enough to x, $f^m(x)$ is near $f^m(x')$; and since the unstable manifolds varies continuously, $W^u(f^m(x))$ is arbitrarily close to $W^u(f^m(x'))$ for x' close enough to x, say for all x' in a neighborhood V_x.

Now, the compactness of M guarantees that there are a finite subcollection of those open sets: $V_{x_1}, \ldots, V_{x_n}$ such that

$$M = \bigcup_{i=1}^{n} V_{x_i},$$

and to each x_i, we have a $m_i = m(x_i)$ associated to it. Since we are stating a property of the unstable manifold, if $W^u(f^m(x)) \cap W^s_{loc}(p) \neq \emptyset$ then $W^u(f^{m'}(x)) \cap W^s_{loc}(p) \neq \emptyset$ for $m' > m$. So, choosing the maximum of those m_i, we obtain an order m that makes the intersection occur for every $x \in M$.

To see that this intersection is actually transverse, we just notice that the ball $B_r(p)$ lies inside a product neighborhood and hence, if a leaf of the unstable foliation $\mathcal{F}^u$, intersects this ball, it must intersect $W^s_{loc}(p)$ and must do it transversely. $\square$

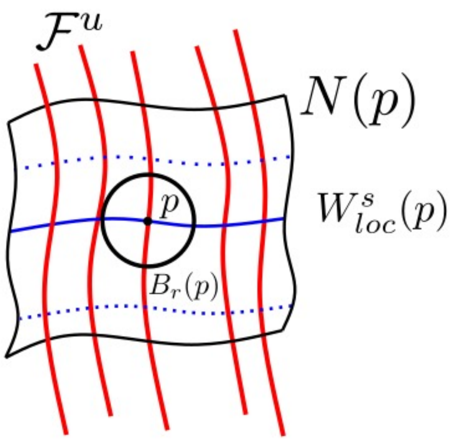

Figure A.3: If a leaf of $\mathcal{F}^u$ intersects the ball $B_r(p)$, it crosses a product neighborhood $N(p)$.

Lemma 27. *There is an integer N with the following property: if $z \in M$ is such that $W^u(z)$ intersects $W^u_{loc}(p)$ transversely, then $f^N(W^u(z))$ is $r-$dense, i.e., for all $y \in M$ and all $r > 0$, there is an integer $N > 0$ such that*

$$f^N(W^u(z)) \cap B_r(y) \neq \emptyset.$$

Proof. Fix a point $y \in M$ and $r > 0$. We are going to find an integer N such that $N > 0$ such that $f^N(W^u(z)) \cap B_r(y) \neq \emptyset$.

From Lemma 26 we know that, for every $x \in M$, the unstable manifold $W^u(x)$ intersects $W^s_{loc}(p)$, the local stable manifold of the fixed point p, transversely: just apply the lemma to $f^{-m}(x)$. For each $x \in M$ consider a small disc D_x transverse to $W^s_{loc}(p)$ lying inside $W^u(x)$.

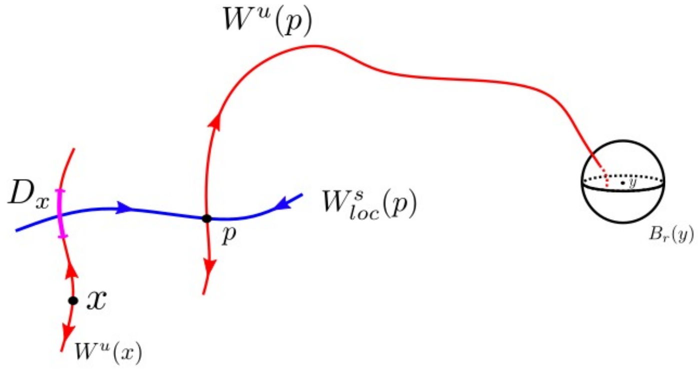

Figure A.4: Applying Lemma 26 to obtain a disc $D_x \subset W^u(x)$ transverse to $W^s_{loc}(p)$.

Applying Corollary 8 to the fixed point p we have $\overline{W^u(p)} = M$. Next, applying the $\lambda-$Lemma (Theorem 25), we obtain an order n_x such that for $n \geq n_x$ we have:

$$W^u(f^{n_x}(x)) \cap B_r(y) \neq \emptyset.$$

Observe that, by continuity of $z \mapsto W^u(f^{n_x}(z))$ in the Hausdorff topology, if $z \in M$ is sufficiently close to x, then $W^u(f^{n_x}(z)) \cap B_r(y) \neq \emptyset$. In other words, there is an open set V_x around x in M such that if $z \in V_x$ then $W^u(f^{n_x}(z)) \cap B_r(y) \neq \emptyset$.

By compactness of M, there is a finite cover of open sets V_{x_i}, $i = 1, \dots, k$, such that $M = \bigcup_{i=1}^{k} V_{x_i}$. Hence, if we set $N = \max\{n_{x_1}, \dots, n_{x_k}\}$, then for all $n \geq N$ we have:

$$W^u\left(f^n(x)\right) \cap B_r(y) \neq \emptyset,$$

for all $x \in M$. This proves the lemma. $\square$

We now apply Lemma 27 to conclude the proof of the theorem. Fix $y \in M$ and $r > 0$ and, to prove the theorem we are going to show that $W^u(x) \cap B_r(y) \neq \emptyset$.

Lemma 27 implies that there is an integer N such that for all $y \in M$ and $r > 0$, we have

$$W^u\left(f^N(x)\right) \cap B_r(y) \neq \emptyset,$$

for all $x \in M$. Since f is a diffeomorphism, this is enough to prove the theorem: instead of x, just consider $f^{-N}(x)$ and we are done. $\square$

As in the case of Anosov flows, we show that if f is a minimal Anosov diffeomorphism, then it is topologically mixing.

Theorem 26. *Let $f\colon M \to M$ be a Anosov diffeomorphism on a compact and connected Riemannian manifold M. If $\overline{W^u(x)} = M$ for all $x \in M$, then f is topologically mixing.*

Proof. As above, the proof will follow from a pair of lemmas (here we follow the ideas contained in [BS02]):

Lemma 28. *If every unstable manifold is dense in M, then for every $\varepsilon > 0$ there is $R = R(\varepsilon) > 0$ such that every ball of radius R in every unstable manifold is $\varepsilon-$dense on M.*

Proof. Let $x \in M$ and notice that $W^u(x) = \bigcup_{R>0} W^u_R(x)$, where $W^u_R(x)$ represents the local unstable manifold of diameter R around x. Since $W^u(x)$ is dense, there is $R(x) > 0$ such that $W^u_{R(x)}(x)$ is $\varepsilon/2-$dense on M. Moreover, since the foliation W^u is continuous, there exists a $\delta(x) > 0$ such that $W^u_{R(x)}(y)$ is $\varepsilon-$dense for all $y \in B_{\delta(x)}(x)$.

Since we are supposing M compact, there is a finite subcollection $\mathcal{B}'$ of the collection $\mathcal{B} := \{B_{\delta(x)}(x) \mid x \in M\}$ that still covers M. By taking R to be the maximum $R(x)$ associated with the balls on $\mathcal{B}'$, we obtain an uniform radius R such that every R-ball in some unstable manifold is dense on M. $\square$

Now let $U, V \subseteq M$ be non-empty open sets and $x \in U$. Inside U, consider $B^u_\delta(x) \subseteq W^u(x) \cap U$ a small disc of unstable manifold; and inside V consider a small ball B_ε of radius ε.

Since D lies inside a unstable foliation, there exists $k \in \mathbb{N}$ such that $\operatorname{diam}(f^m(D)) > 2R$ for all $m \geq k$. Hence, by Lemma 28 above, $f^m(D)$ is $\varepsilon-$dense on M, for all $m \geq k$. Therefore, $f^m(D) \cap B_\varepsilon \neq \emptyset$ and, in particular, $f^m(U) \cap V \neq \emptyset$, again for all $m \geq k$. This proves that f is topologically mixing. $\square$

APPENDIX **B**

Approximation by rationals

We dedicate this Appendix to the proof of number theoretic lemma inside of Lemma 13, i.e.,

Lemma. *For all* $\lambda, t_1, \ldots, t_N \in (0, +\infty)$ *and* $n_0 \in \mathbb{N}$, *there are* $n_1, \ldots, n_N \geq n_0$ *and* $t \in \mathbb{R}$ *such that*

$$|n_i t_i - t| < \lambda t_i,$$

for all $i = 1, \ldots, N$.

To do so, we follow the proof given in [Niv05], where the above lemma is obtained as a corollary of a pair of theorems. Throughout this Appendix B, we shall denote by $\lfloor \alpha \rfloor$ the largest integer that is not larger than the real number α. Alternatively, for each α the number $\lfloor \alpha \rfloor$ is the only solution m in $\mathbb{Z}$ for the inequalities:

$$m \leq \alpha < m + 1.$$

Also, we denote by $\lceil \alpha \rceil$ the least integer not less than α, i.e., the unique solution $m \in \mathbb{Z}$ to the inequality

$$m < \alpha \leq m + 1.$$

For a real number α, we have $-\lfloor -\tau \rfloor = \lceil \tau \rceil$.

A classical problem on Number Theory asks how well an irrational number α can be approximated by a rational number. More precisely: given an irrational number α and $\varepsilon > 0$, are there integers k and h such that $|k\alpha - h| < \varepsilon$? The answer is yes and is given by the following theorem:

Theorem. *Given any irrational number* α *and any positive integer* n*, there exist integers* h *and* k *with* $0 < k \leq n$ *such that*

$$|k\alpha - h| < \frac{1}{n}.$$

Proof. A proof of this fact can be found in [Niv05] itself, at p.44, Theorem 4.2; but also in many books on Number Theory, such [Fig11] (see *Teorema 5.2*, p. 22) and [MAR+15] (*Exemplo 0.10*, p. 12). □

There are several ways to generalize the above theorem for higher dimensions. One goes as follows: given real numbers α_1 and α_2, can we find a lattice point $k = (k_1, k_2) \in \mathbb{Z}^2$ such that both $k_1\alpha_1$ and $k_2\alpha_2$ are arbitrarily close to integers? The following theorems provide that we still have positive answers for this kind of question.

Theorem 27. *Let $A \in \mathcal{M}_{n\times m}(\mathbb{R})$ be a $n \times m$ real matrix with entries denoted by a_{ij} and let τ be a real number greater or equal to 1. Also, define $T = -\lfloor -\tau \rfloor = \lceil \tau \rceil$, so that T is the smallest integer not less than τ.*

Then there exist lattice points $(k_1, \ldots, k_m) \in \mathbb{Z}^m$ and $(h_1, \ldots, h_n) \in \mathbb{Z}^n$ such that:

- $|k_j| \leq T^{n/m}$ *for all $j = 1, \ldots, m$;*
- $\sum_{j=1}^m |k_j| \neq 0$*;*
- $\left|\sum_{j=1}^m a_{ij}k_j - h_i\right| < 1/\tau$*, for $i = 1, \ldots, n$.*

This theorem can be interpreted as follows: given n linear forms,

$$\Psi_i(k_1, \ldots, k_m) = \sum_{j=1}^m a_{ij}k_j, \quad \text{where } i = 1, \ldots, n,$$

the theorem states that we can chose $(k_1, \ldots, k_m) \in \mathbb{Z}^m$ in such a way that this n forms are arbitrarily close to an integer. To put it in another way: by setting

$$\Psi(k_1, \ldots, k_m) = (\Psi_1(k_1, \ldots, k_m), \ldots, \Psi_n(k_1, \ldots, k_m)),$$

and thinking of it as a point in $\mathbb{R}^n$, the theorem states that fixing an arbitrary distance, we can choose $(k_1, \ldots, k_m)$ properly in such a way that $\Psi(k_1, \ldots, k_m)$ will at most this distance far from some point $(h_1, \ldots, h_n) \in \mathbb{Z}^n$ with integer coordinates in $\mathbb{R}^n$.

Since we're requesting *some* point $(h_1, \ldots, h_n) \in \mathbb{Z}^n$, this goal would be trivially satisfied by setting $(k_1, \ldots, k_m) = (0, \ldots, 0)$. This justifies the restriction $\sum_{j=1}^m |k_j| \neq 0$ in the conclusion: the theorem provides a non-trivial point satisfying the required condition.

Proof of Theorem 27. Fixed a positive integer q, there are $(q+1)^m$ points $(y_1, \ldots, y_m)$ in $\mathbb{Z}^m$ such that $0 \leq y_i \leq q$, for $i = 1, \ldots, m$. If we set

$$\omega_i = \sum_{j=1}^m a_{ij}y_j$$

for each $i = 1, \ldots, n$, there are also $(q+1)^m$ options for the $n-$tuple $(\omega_1, \ldots, \omega_n) \in \mathbb{Z}^n$.

Now, for each $i = 1, \ldots, n$, let x_i be the integer that $0 \leq x_i - \omega_i < 1$, i.e., for each $i = 1, \ldots, n$, define x_i to be $x_i = \lceil \omega_i \rceil$. By doing this we obtain a collection $\mathcal{Q}$ of $(q+1)^m$ points in the cube $\mathcal{C} := [0,1)^n$. Partitioning this cube $\mathcal{C}$ into T^n smaller cubes of side $\frac{1}{T}$, with sides being parallel hyperplanes and such that the projection onto each side of $\mathcal{C}$ being half-open such as $[0, 1/T)^n$.

Next, set $q = \lfloor T^{n/m} \rfloor$, so that:

$$(q+1)^m = (\lfloor T^{n/m} \rfloor + 1)^m > \left(T^{n/m}\right)^m = T^n.$$

Hence, by the pigeon-hole principle, the $(q+1)^m$ points in $\mathcal{Q}$ being distributed in T^n cubes of side $1/T$, cannot all lie in different cubes. Thus, there are at least two different points of $\mathcal{Q}$ lying in the same subcube, say $(x_1 - \omega_1, \ldots, x_n - \omega_n)$ and $(x'_1 - \omega'_1, \ldots, x'_n - \omega'_n)$, where $\omega'_i = \sum_{j=1}^m a_{ij}y'_i$ for some lattice point

$(y_1', \ldots, y_m')$ in $\mathbb{Z}^m$, different from $(y_1, \ldots, y_m)$, but still with $0 \le y_i' \le q$ for all $i = 1, \ldots, m$. Then,

$$\begin{aligned}\frac{1}{T} &> \left|(x_i - \omega_i) - (x_i' - \omega_i')\right| \\ &= \left|(\omega_i' - \omega_i) - (x_i' - x_i)\right| \\ &= \left|\left(\sum_{j=1}^{m} a_{ij} y_j' - \sum_{j=1}^{m} a_{ij} y_j\right) - (x_i' - x_i)\right| \\ &= \left|\sum_{j=1}^{m} a_{ij}\left(y_j' - y_j\right) - (x_i' - x_i)\right|,\end{aligned}$$

for $i = 1, \ldots, n$. So, if we fix $k_j = y_j' - y_j$ and $h_i = x_i' - x_i$, and use the fact that $\tau < T$, we obtain: $\left|\sum_{j=1}^{m} a_{ij} k_j - h_i\right| < 1/\tau$, for $i = 1, \ldots, n$.

In order to obtain the other two conclusions of the theorem first observe that, being $(y_1, \ldots, y_m)$ and $(y_1', \ldots, y_m')$ distinct, there is at least one index $j_0 \in \{1, \ldots, m\}$ such that $y_{j_0} \neq y_{j_0}'$. So, there exists j_0 such that $k_{j_0} \neq 0$ and then $\sum_{j=1}^{m} |k_j| \neq 0$.

Finally, since $0 \le y_j \le q$ and $0 \le y_j' \le q$ for all $j = 1, \ldots, m$, it follows that

$$|k_j| = |y_j' - y_j| \le q = \lfloor T^{n/m} \rfloor \le T^{n/m},$$

as desired. □

Corollary. *Given any real numbers $\alpha_1, \ldots, \alpha_m$ and any integer $t \ge 1$, there exists a lattice point $(k_1, \ldots, k_m, h)$ with $|k_j| \le t$ for all $j = 1, \ldots, m$ and $\sum |k_j| \neq 0$, such that*

$$\left|\sum_{j=1}^{m} \alpha_j k_j - h\right| < 1/t^m.$$

Proof. In the theorem above, substitute n by 1 to get only one row in the matrix A,

$$\alpha_{11}, \ldots, \alpha_{1m},$$

and we call α_{1j} by α_j, for each $j = 1, \ldots, m$. Still in the theorem above, replace τ by t^m to obtain $T = t^m$ and $T^{n/m} = T^{1/m} = t$.

Hence, the conclusions of the theorem will be: there exists $h_1 \in \mathbb{Z}$, that we call h, and $(k_1, \ldots, k_m) \in \mathbb{Z}^m$ and $h \in \mathbb{Z}$ with $|k_j| \le t$ for all $j = 1, \ldots, m$, $\sum_{j=1}^{m} |k_j| \neq 0$ and

$$\left|\sum_{j=1}^{m} \alpha_j k_j - h\right| < 1/t^m.$$

□

The next consequence of Theorem 27 we present here is the central fact that we use to prove the statement inside Lemma 13:

Theorem 28. *Given any real numbers $\alpha_1, \ldots, \alpha_n$, there exists infinitely many sets of integers $k, q_1, \ldots, q_n$, with $k > 0$, such that*

$$\left|\alpha_i - \frac{q_i}{k}\right| < \frac{1}{k\sqrt[n]{k}}, \tag{B.1}$$

for $i = 1, \ldots, n$.

Proof. If, in Theorem 27, we set $m = 1$, substitute α_{i1} by α_i, and demand τ to be a positive integer, in such a way that $\tau = \lceil \tau \rceil = T$, we conclude that, for any positive integer T there is a lattice point $(k_1, h_1, h_2, \dots, h_n) \in \mathbb{Z}^{n+1}$, with $0 < |k_1| \leq T^n$, and such that:

$$|\alpha_i k_1 - h_i| < \frac{1}{T},$$

for $i = 1, \dots, n$. Note that each of the n inequalities still holds if we exchange $(k_1, h_1, h_2, \dots, h_n)$ by $(-k_1, -h_1, -h_2, \dots, -h_n)$. Either k_1 or $-k_1$ must be positive. Call it k and call the point the it is a coordinate by $(k, q_1, \dots, q_n)$. In particular, we have shown that for each positive integer T there is a lattice point $(k, q_1, \dots, q_n) \in \mathbb{Z}^{n+1}$ such that $0 < k < T^n$ and

$$|\alpha_i k - q_i| < \frac{1}{T}, \tag{B.2}$$

for $i = 1, \dots, n$.

Moreover, since $0 < k < T^n$, we have that $\frac{1}{T} < \frac{1}{\sqrt[n]{k}}$. Then, $|\alpha_i k - q_i| < \frac{1}{T}$ if and only if $\left|\alpha_i - \frac{q_i}{k}\right| < \frac{1}{kT}$, and then

$$\left|\alpha_i - \frac{q_i}{k}\right| < \frac{1}{kT} < \frac{1}{k\sqrt[n]{k}},$$

i.e., we proved that there exists $(k, q_1, \dots, q_n) \in \mathbb{Z}^{n+1}$ such that $0 < k < T^n$ and that $\left|\alpha_i - \frac{q_i}{k}\right| < \frac{1}{k\sqrt[n]{k}}$, for $i = 1, \dots, n$.

To conclude the proof, we need to prove that there exist infinitely many lattice points such as the one above. To do so, we consider two distinct cases: one is what happens when all the real numbers $\alpha_1, \dots, \alpha_n$ are rational; the second is when at one of them is irrational.

In the first case, i.e., when $\alpha_1, \dots, \alpha_n \in \mathbb{Q}$, it is easy to find $(k, q_1, \dots, q_n) \in \mathbb{Z}^{n+1}$ satisfying the inequalities stated. Actually, we can ignore all that was done until now and consider k to be any integer that is a common multiple of the denominators of all α_i's and take $q_i = k\alpha_i$. Of course, there are infinitely many choices for k.

So we remain in the last case, i.e., some of the α_i's is irrational. Suppose, without loss of generality, that $\alpha_1 \in \mathbb{R} \setminus \mathbb{Q}$. To obtain a contradiction, suppose there is only a finite number of integers $k, q_1, \dots, q_n$ such that $\left|\alpha_1 - \frac{q_i}{k}\right| < \frac{1}{k\sqrt[n]{k}}$, for $i = 1, \dots, n$. In particular, there would be only a finite number of corresponding values $|\alpha_1 k - q_1|$, all positive (since α_1 is irrational). But each of these values would exceed $\frac{1}{T}$, provided we choose T to be large enough. Now, this large enough T would, by the argumentation that we have done to obtain the inequality $(B.2)$, give us a different set of integers $k, q_1, \dots, q_n$. This is a contradiction since we were supposing the first set of integers we have picked was the largest as possible. This completes the proof of the theorem. □

Finally, we prove the result inside Lemma 13:

Lemma. *For all* $\lambda, t_1, \dots, t_N \in (0, +\infty)$ *and* $n_0 \in \mathbb{N}$, *there are* $n_1, \dots, n_N \geq n_0$ *and* $t \in \mathbb{R}$ *such that*

$$|n_i t_i - t| < \lambda t_i,$$

for all $i = 1, \dots, N$.

Proof. First we prove that, for all $\varepsilon > 0$ and all $t_1, \dots, t_N > 0$, the set of $N+1-$tuples $(t, n_1, \dots, n_N) \in \mathbb{R} \times \mathbb{Z}^N$ that satisfy

$$\left|\frac{t}{t_i} - n_i\right| < \varepsilon,$$

for each $i = 1, \ldots, N$ is infinite.

To prove this claim, fix $\alpha_i = \frac{1}{t_i}$ for each $i = 1, \ldots, N$ in Theorem 28. It guarantees the existence of infinitely many $N+1-$tuples $(t, n_1, \ldots, n_N) \in \mathbb{R} \times \mathbb{Z}^N$, with $t > 0$ and satisfying inequality (B.1), i.e., $\left|\alpha_i - \frac{n_i}{t}\right| < \frac{1}{t\sqrt[N]{t}}$, for each $i = 1, \ldots, N$. In other words,

$$|t\alpha_i - n_i| = t \cdot \left|\alpha_i - \frac{n_i}{t}\right| < \frac{1}{\sqrt[N]{t}},$$

for each $i = 1, \ldots, N$. Since $t\alpha_i = \frac{t}{t_i}$ and since there are infinitely many $(t, n_1, \ldots, n_N)$ that make this inequality work, we may choose $t > 0$ sufficiently large such that $\frac{1}{\sqrt[N]{t}} < \varepsilon$, and hence the first statement is proved.

Moreover, if we ask for $\varepsilon > 0$ to be sufficiently small, since all the t_i's and t are positive, we may ask that the n_i's are all positive as well.

Now, how do we move to the statement of the lemma? First, choose $\varepsilon > 0$ to be $\varepsilon < \lambda$. So, for every $i = 1, \ldots, N$, we find infinitely many $N+1-$tuples $(t, n_1, \ldots, n_N)$ such that $t > 0$, the n_i's are positive (by reducing ε if necessary) and since there are infinitely many of then, we can ask them all to be greater than $n_0 \in \mathbb{N}$ fixed. Finally, each n_i satisfy:

$$|n_i t_i - t| < \varepsilon t_i < \lambda t_i,$$

for all $i = 1, \ldots, N$. $\square$

Bibliography

[Alv13] S. Alvarez. “Mesures de Gibbs et mesures harmoniques pour les feuilletages aux feuilles courbées négativement”. PhD thesis. Université de Bourgogne, 2013.

[AM08] R. Abraham and J. E. Marsden. *Foundations of mechanics*. 364. American Mathematical Soc., 2008.

[Ano67] D. V. Anosov. “Geodesic flows on closed Riemannian manifolds of negative curvature”. In: *Trudy Matematicheskogo Instituta Imeni VA Steklova* 90 (1967), pp. 3–210.

[Arn13] V. I. Arnol’d. *Mathematical methods of classical mechanics*. Vol. 60. Springer Science & Business Media, 2013.

[BS02] M. Brin and G. Stuck. *Introduction to dynamical systems*. Cambridge University Press, 2002.

[CN13] C. Camacho and A. L. Neto. *Geometric theory of foliations*. Springer Science & Business Media, 2013.

[Cou09] Y. Coudène. “A short proof of the unique ergodicity of horocyclic flows”. In: *Ergodic theory* 485 (2009), pp. 85–89.

[Dal10] F. Dal’Bo. *Geodesic and horocyclic trajectories*. Springer Science & Business Media, 2010.

[Day20] J. J. Day. “Topologically Mixing Suspension Flows”. MA thesis. Brigham Young University, 2020.

[DF92] M. P. Do Carmo and J. Flaherty Francis. *Riemannian geometry*. Vol. 6. Springer, 1992.

[DP03] V. J. Donnay and C. C. Pugh. “Anosov geodesic flows for embedded surfaces”. In: *Astérisque* 287 (2003), pp. 61–69.

[FH19] T. Fisher and B. Hasselblatt. *Hyperbolic flows*. European Mathematical Society, 2019.

[Fig11] D. G. Figueiredo. *Números irracionais e transcendentes. 3a edição*. SBM, 2011.

[Fur73] H. Furstenberg. “The unique ergodigity of the horocycle flow”. In: *Recent advances in topological dynamics*. Springer, 1973, pp. 95–115.

[FW80] J. Franks and B. Williams. “Anomalous Anosov flows”. In: *Global theory of dynamical systems*. Springer, 1980, pp. 158–174.

[Hed36] G. A. Hedlund. “Fuchsian groups and transitive horocycles”. In: *Duke Mathematical Journal* 2.3 (1936), pp. 530–542.

[KH97] A. Katok and B. Hasselblatt. *Introduction to the modern theory of dynamical systems*. 54. Cambridge University Press, 1997.

[Kli74] W. Klingenberg. “Riemannian manifolds with geodesic flow of Anosov type”. In: *Annals of Mathematics* 99.1 (1974), pp. 1–13.

[Kni02] G. Knieper. “Hyperbolic dynamics and Riemannian geometry”. In: *Handbook of dynamical systems*. Vol. 1. Elsevier, 2002, pp. 453–545.

[Lee13] J. M. Lee. “Smooth manifolds”. In: *Introduction to Smooth Manifolds*. Springer, 2013, pp. 1–31.

[MAR+15] F. MARTINEZ et al. *Teoria dos Números: um passeio com primos e outros números familiares pelo mundo inteiro*. Instituto de Matemática Pura e Aplicada, 2015.

[MR20] Í. Melo and S. Romaña. “Contributions to the study of Anosov geodesic flows in non-compact manifolds”. In: *Discrete & Continuous Dynamical Systems* 40.9 (2020), p. 5149.

[Niv05] I. Niven. *Irrational numbers*. 11. Cambridge University Press, 2005.

[Pat12] G. P. Paternain. *Geodesic flows*. Vol. 180. Springer Science & Business Media, 2012.

[PD12] J. J. Palis and W. De Melo. *Geometric theory of dynamical systems: an introduction*. Springer Science & Business Media, 2012.

[Pla72] J. F. Plante. “Anosov flows”. In: *American Journal of Mathematics* 94.3 (1972), pp. 729–754.

[PSW97] C. Pugh, M. Shub, and A. Wilkinson. “Hölder foliations”. In: *Duke Mathematical Journal* 86.3 (1997), pp. 517–546.

[Ulc21] C. Ulcigrai. “Slow chaos in surface flows”. In: *Bollettino dell’Unione Matematica Italiana* 14.1 (2021), pp. 231–255.

[VO16] M. Viana and K. Oliveira. *Foundations of ergodic theory*. Cambridge Studies in Mathematics 151. Cambridge University Press, 2016.

[Wen16] L. Wen. *Differentiable dynamical systems*. Vol. 173. American Mathematical Soc., 2016.