Conley's Theorem

Odylo Abdalla Costa

September 15, 2021

Contents

1 Introduction

These notes intend to be a guide to a seminar on a reading course led by professor Lorenzo Díaz at PUC-RJ on the second semester of 2021. Of course, these notes may contain errors, if you find some, please send me at `odylocosta@gmail.com`. I would appreciate it!

Here, I follow [1], [3] and [4] to present a proof of Conley's Theorem (a.k.a., the Fundamental Theorem of Dynamical Systems). Another great reference is [2]. To give the statements and proofs for the general theory of $\varepsilon-$chains on compact metric spaces I have followed mainly [3] and lecture notes written by professor Andres Koropecki for a course lectured at UFF by him. The lecture notes can be found **here**.

Conley's Theorem states that every dynamical system on a compact metric space decomposes the space into a chain recurrent part and a gradient-like one. This very general statement and its relation with fundamental topics on Dynamical Systems are some of the reasons that make this theorem be called the Fundamental Theorem of Dynamical Systems. A more technical and precise statement is:

Theorem (Conley)**.** *Every homeomorphism $f\colon X \to X$ on a compact metric space X admits a complete Lyapunov function, i.e., a continuous function $\Phi\colon X \to \mathbb{R}$ satisfying:*

(i) for all $x \notin \mathcal{CR}(f)$, $\Phi(f(x)) < \Phi(x)$;

(ii) for every $x, y \in \mathcal{CR}(f)$, $\Phi(x) = \Phi(y)$ if and only if $x \sim y$;

(iii) $\Phi(\mathcal{CR}(f))$ is a compact nowhere dense subset of $\mathbb{R}$.

2 Chain classes

The first thing we do here is to remember what are chain classes. To do so, fix a homeomorphism $f\colon X \to X$ defined on a compact metric space (X, d).

Definition 1. *Given $\varepsilon > 0$ and two points x and y on X, an $\varepsilon-$**chain from** x **to** y is a sequence $x_0, x_1, \dots, x_{k-1}, x_k$ of points in X, with $k > 1$, such that $x = x_0$, $x_k = y$ and*

$$d(f(x_i), x_{i+1}) < \varepsilon,$$

*for all $i = 0, \dots, k-1$. If there is an $\varepsilon-$chain from x to y we write $x \xrightarrow{\varepsilon} y$, and if $x = y$ we say that the $\varepsilon-$chain is **periodic**.*

Definition 2. *A point $x \in X$ is said to be **chain recurrent** if for every $\varepsilon > 0$, there exists an $\varepsilon-$chain from x to x, i.e., $x \xrightarrow{\varepsilon} x$. We represent the set of all chain recurrent points of a dynamical system f as $\mathcal{CR}(f)$.*

In the context we are at, i.e., where $f\colon X \to X$ is an homeomorphism on a compact metric space (X, d), we have the following result:

Proposition 1. *the set $\mathcal{CR}(f)$ is non-empty, compact, invariant, and*

$$\overline{\mathrm{Per}\,(f)} \subseteq \mathrm{L}\,(f) \subseteq \Omega(f) \subseteq \mathcal{CR}(f).$$

Proof. The inclusion $\overline{\mathrm{Per}\,(f)} \subseteq \mathrm{L}\,(f)$ is straightforward.

To see that $\mathrm{L}\,(f) \subseteq \Omega(f)$, observe that, being $\Omega\,(f)$ a closed set, it suffices to show that, for $x \in X$, we have $\omega_f\,(x) \subseteq \Omega\,(f)$ and $\alpha_f\,(x) \subseteq \Omega\,(f)$. Moreover, being f an homeomorphism, $\alpha_f\,(x) = \omega_{f^{-1}}\,(x)$ and $\Omega\,(f) = \Omega\,(f^{-1})$.

Therefore, to prove the inclusion, we only need to prove that, given $x \in X$ then $\omega_f\,(x) \subseteq \Omega\,(f)$.

Suppose that $y \in \omega_f\,(x)$, i.e., that there exists $n_k \to +\infty$ such that $f^{n_k}\,(x) \to y$. Let U to be a neighborhood of y. There exists an increasing subsequence $(n_k)_k$, say n_{k_j}, and an order $J \in \mathbb{N}$ such that for all $j \geq J$, $f^{n_{k_j}}\,(x) \in U$. By calling $z = f^{n_{k_J}}\,(x)$ and $n = n_{k_{J+1}} - n_{k_J}$, we have:

$$z \in U \text{ e } f^n\,(z) = f^{n_{k_{J+1}}}\,(x) \in U.$$

So we have proven that $\omega_f\,(x) \subseteq \Omega\,(f)$ for all $x \in X$. From what we have said before, this shows $L\,(f) \subseteq \Omega\,(f)$.

Finally, for $\Omega(f) \subseteq \mathcal{CR}(f)$, fix $x \in \Omega(f)$ and $\varepsilon > 0$. We need to prove that $x \xrightarrow{\varepsilon} x$. Let $\delta > 0$ such that $d(f(x), f(y)) < \varepsilon$ if $d(x, y) < \delta$. Suppose $\delta < \varepsilon$. Then the neighborhood $B(x, \delta)$ is nonwandering, i.e., there is $n > 0$ such that $f^n(B(x, \delta)) \cap B(x, \delta) \neq \emptyset$. In particular, there is $y \in X$ such that $d(x, y) < \delta$ and $d(f^n(y), x) < \delta$.

We claim that $x = x_0, f(y), f^2(y), \dots, f^{n-1}(y), x$ is an $\varepsilon-$chain. Indeed, by our choice for $\delta > 0$, we have $d(f(x), f(y)) < \varepsilon$, and $d(f^{n-1}(y), x) = d(f^n(y), x) < \delta < \varepsilon$. Hence $x \xrightarrow{\varepsilon} x$ and $x \in \mathcal{CR}(f)$. $\square$

Next we present a notion of open ball on the setting of $\varepsilon-$chains.

Proposition 2. *Fix $\varepsilon > 0$ and consider the set $\Omega(x, \varepsilon) = \{y \in X \mid x \xrightarrow{\varepsilon} y\}$. Then,* *(i)* $\Omega(x,\varepsilon)$ *is open;*

(ii) $B(f(\Omega(x,\varepsilon)),\varepsilon) := \bigcup_{y\in f(\Omega(x,\varepsilon))} B(y,\varepsilon) \subset \Omega(x,\varepsilon)$*;*

(iii) for all $y \in \overline{\Omega(x,\varepsilon)}$*, we have* $\Omega(y,\varepsilon) \subset \Omega(x,\varepsilon)$*.*

(iv) $f(\overline{\Omega(x,\varepsilon)}) \subseteq \Omega(x,\varepsilon)$

Proof. To see $\Omega(x,\varepsilon)$ is open notice that if $y \in \Omega(x,\varepsilon)$, then there is an $\varepsilon-$chain from x to y, say $x = x_0, x_1, \ldots, x_n = y$. Set $\delta = d(f(x_{n-1}), y)$. Hence, $0 < \delta < \varepsilon$ and, by exchanging x_n by any y' such that $d(y,y') < \varepsilon - \delta$ we still have a $\varepsilon-$chain. Hence, $B(y, \varepsilon - \delta) \subseteq \Omega(x,\varepsilon)$, proving $\Omega(x,\varepsilon)$ is open.

For item *(ii)*, take $y \in \Omega(x,\varepsilon)$ and an $\varepsilon-$chain from x to y. By adding a point $z \in B(f(y),\varepsilon)$ at the end of the chain, we obtain an $\varepsilon-$chain from x to z. Therefore, $B(f(y),\varepsilon) \subset \Omega(x,\varepsilon)$. This proves that $B(f(\Omega(x,\varepsilon)),\varepsilon) \subset \Omega(x,\varepsilon)$.

Lastly, since for every subset $Y \subseteq X$, $B(\overline{Y},\varepsilon) = B(Y,\varepsilon)$, item *(ii)* implies

$$B(\overline{f(\Omega(x,\varepsilon))},\varepsilon) \subset \Omega(x,\varepsilon).$$

So, if $y \in \overline{\Omega(x,\varepsilon)}$ we have $f(y) \in f(\overline{\Omega(x,\varepsilon)}) \subset \overline{f(\Omega(x,\varepsilon))}$ and hence $B(f(y),\varepsilon) \subseteq \Omega(x,\varepsilon)$. In particular, this proves item *(iv)*.

With that in hands, suppose by contradiction that $\Omega(y,\varepsilon) \subset \Omega(x,\varepsilon)$ doesn't happen. Then there is an $\varepsilon-$chain $y = x_0, x_1, \ldots, x_n$ with $x_n \notin \Omega(x,\varepsilon)$. Since $x_1 \in B(f(y),\varepsilon)$, the previous paragraph implies $x_1 \in \Omega(x,\varepsilon)$. Supposing moreover that $i = n > 1$ is the smaller $i > 1$ such that $x_i \notin \Omega(x,\varepsilon)$ (if that's not the case we decrease n). Then, $x_{n-1} \in \Omega(x,\varepsilon)$ and, from item *(ii)*, $B(f(x_{n-1}),\varepsilon) \subseteq \Omega(x,\varepsilon)$. Since $d(f(x_{n-1}),x_n) < \varepsilon$, we conclude that $x_n \in \Omega(x,\varepsilon)$, contradicting the choice of n. □

As we claimed in Section 1, we wish to break the compact set X into a disjoint union of indecomposable compact invariant sets. Let's precise what we mean by **indecomposable**:

Definition 3. *A compact invariant set* $\Lambda \subset X$ *is called* ***indecomposable*** *if* Λ *cannot be written as a union of two non-empty compact invariant sets.*

To decompose the whole space X may be too much to ask for an arbitrary dynamical system $f\colon X \to X$. However, its chain recurrent set $\mathcal{CR}(f)$ can itself be decomposed by a equivalence relation that will be extremely important for what comes next.

Definition 4. *Given two points* x *and* y *in* $\mathcal{CR}(f)$*, we say that* x *and* y *are* ***chain related****, and write* $x \sim y$*, if for any* $\varepsilon > 0$ *there exists an* $\varepsilon-$*chain from* x *to* y *and* $\varepsilon-$*chain from* y *to* x*. Each equivalence class is called* ***chain class of*** f*.*

Proposition 3. *The relation described above is indeed an equivalence relation on* $\mathcal{CR}(f)$ *and each chain class is compact and invariant.*

Proof. First we check that $\sim$ is indeed an equivalence relation. To do so, let x, y and z points in $\mathcal{CR}(f)$ and $\varepsilon > 0$. Notice that both reflexive and symmetric proprieties follow from the definition of the relation $\sim$ on $\mathcal{CR}(f)$.

For transitivity suppose there are an $\varepsilon-$chain $x_0, \ldots, x_n$ from x to y and an $\varepsilon-$chain $y_0, \ldots, y_{m-1}$ from y to z, then there is an $\varepsilon-$chain from x to z. Indeed, let $z_0 = x_0 =$

$x, \ldots, z_n = x_n$ and $z_{n+1} = y_0, \ldots, z_{n+m} = y_{m-1} = z$, then for all $i = 0, \ldots, n+m$ we have $d(f(z_i), z_{i+1}) < \varepsilon$. Hence, $\sim$ is an equivalence relation in $\mathcal{CR}(f)$.

Now, we prove that $f(\mathcal{CR}(f)) = \mathcal{CR}(f)$. First we check that $f(\mathcal{CR}(f)) \subseteq \mathcal{CR}(f)$. To do so, let $x \in \mathcal{CR}(f)$ and $\varepsilon > 0$, and we prove that $f(x) \xrightarrow{\varepsilon} f(x)$.

By continuity of f at $f(x)$, there is $\delta > 0$ such that for all $y \in X$ with $d(y, f(x)) < \delta$ one has $d(f(y), f^2(x)) < \varepsilon/2$. By reducing $\delta > 0$ we can ask it to be $0 < \delta < \varepsilon/2$, so if $x = x_0, x_1, \ldots, x_n = x$ is a $\delta-$chain, then the sequence $f(x), x_1, \ldots, x_{n-1}, x, f(x)$ is a $\delta-$chain because:

$$d(f(f(x)), x_2) \leq d(f(f(x)), f(x_1)) + d(f(x_1), x_2) < \varepsilon/2 + \delta < \varepsilon,$$

since $d(f(x), x_1) < \delta < \varepsilon/2$. Hence, $f(x) \xrightarrow{\varepsilon} f(x)$, proving that $f(\mathcal{CR}(f)) = \mathcal{CR}(f)$.

Reciprocally, let $x \in \mathcal{CR}(f)$ and we prove that $x \in f(\mathcal{CR}(f))$. Since, by continuity of f^{-1} at $x \in X$, given $\varepsilon > 0$ there is $\delta > 0$ (and we ask also that $\delta < \varepsilon/2$) such that, if $d(y, x) < \delta$, then $d(f^{-1}(x), f^{-1}(y)) < \varepsilon/2$. If $x = x_0, x_1, \ldots, x_n = x$ is a $\delta-$chain from x to x, then:

$$f^{-1}(x) = f^{-1}(x_0), x_0, x_1, \ldots, x_{n-2}, f^{-1}(x_n) = f^{-1}(x)$$

is an $\varepsilon-$chain from $f^{-1}(x)$ to $f^{-1}(x)$, since

$$d(f(x_{n-2}), f^{-1}(x)) \leq d(f(x_{n-1}), x_{n-1}) + d(f^{-1}(f(x_{n-1})), f^{-1}(x)) \leq \delta + \varepsilon/2 < \varepsilon,$$

because $d(f(x_{n-1}), x) = d(f(x_{n-1}), x_n) < \delta$. Therefore, $f^{-1}(x) \in \mathcal{CR}(f)$ and we have shown that $f^{-1}(\mathcal{CR}(f)) \subseteq \mathcal{CR}(f)$, or equivalently, $\mathcal{CR}(f) \subseteq f(\mathcal{CR}(f))$, as we claimed.

Finally, we show that $\mathcal{CR}(f)$ is closed (and since X is compact, $\mathcal{CR}(f)$ is also compact). Let $y \in X$ be a point that is accumulated by points in $\mathcal{CR}(f)$. We need to show that $y \in \mathcal{CR}(f)$. Fix $\varepsilon > 0$, by continuity of f at y, there is $\delta > 0$ such that if $d(x, y) < \delta$, then $d(f(x), f(y)) < \varepsilon/2$. Also suppose $\delta < \varepsilon/2$. Now, since y is accumulated by points in $\mathcal{CR}(f)$, there is $x \in \mathcal{CR}(f)$ such that $d(x, y) < \delta$. Since $x \in \mathcal{CR}(f)$, there is a $\delta-$chain from x to x, say $x = x_0, x_1, \ldots, x_n = x$. We claim that

$$y, x_1, \ldots, x_{n-1} = y$$

is an $\varepsilon-$chain from y to y. Indeed, for all $i = 1, \ldots, n-2$, $d(f(x_i), x_{i+1}) < \delta < \varepsilon$ and for $i = 0$ or $i = n$, we have:

$$\begin{aligned} d(f(y), x_1) &\leq d(f(y), f(x)) + d(f(x), x_1) < \varepsilon/2 + \delta < \varepsilon, \\ d(f(x_{n-1}), y) &\leq d(f(x_{n-1}), x) + d(x, y) < \delta + \delta < \varepsilon. \end{aligned}$$

This shows that $y \xrightarrow{\varepsilon} y$ and, since $\varepsilon > 0$ was arbitrary, $y \in \mathcal{CR}(f)$, as we wanted. So, $\mathcal{CR}(f)$ is closed.

□

We finish this Section 2 with a theorem that proves that, indeed, each chain class is indecomposable.

Theorem 1. *Let C be a chain class of f. Then,*

(i) for any $\varepsilon > 0$ there is a $\delta > 0$ such that, for any $x \in C$, any periodic $\delta-$chain through x is contained in the $\varepsilon-$neighborhood $B(C, \varepsilon)$ of C;

(*ii*) C *is indecomposable.*

Proof. To prove the first part, suppose there is some $\varepsilon_0 > 0$ such that, for every $n \geq 1$, there exists $x_0^n \in C$ and a periodic $1/n-$chain $x_0^n, \ldots, x_{j_n}^n$ such that $x_{k_n}^n \notin B(C, \varepsilon_0)$ for some $0 \leq k_n \leq j_n$. Passing to subsequences if necessary, we may assume that $x_0^n \to x$ and that $x_{k_n}^n \to y$. In one hand, x and y are chain related. On the other, $x \in C$ and $y \notin C$, a contradiction. This proves item (i).

Suppose now that C decomposes into a disjoint union of two nonempty compact invariant sets C_1 and C_2. We use item (i) to obtain a contradiction.

In order to do that, take $\varepsilon > 0$ small enough such that

$$B(C_1, \varepsilon) \cap B(C_2, \varepsilon) = \emptyset, \ f\left(B(C_1, \varepsilon)\right) \cap B\left(B(C_2, \varepsilon), \varepsilon\right) = \emptyset.$$

The second inequality means that no point in one iterate of f to an $\varepsilon-$neighborhood of $B(C_2, \varepsilon)$.

Let $x \in C_1$. By item (i), there is a $\delta > 0$ such that any periodic $\delta-$chain through x contained in $B(C_1, \varepsilon) \cup B(C_2, \varepsilon)$. Suppose $\delta < \varepsilon$ and notice that, since C_1 and C_2 are in the same chain class, there is a $\delta-$chain through x the intersects $B(C_2, \varepsilon)$. Hence, there exists $z \in B(C_1, \varepsilon)$ such that

$$f(z) \in B\left(B(C_2, \varepsilon), \delta\right) \subset B\left(B(C_2, \varepsilon), \varepsilon\right),$$

contradicting the choice of $\varepsilon > 0$ and proving item (ii).

□

3 From chain classes to trapping regions

Definition 5. *An open set $U \subset X$ is called a* ***trapping region of*** f *if $f(\overline{U}) \subset U$. A compact invariant set $\Lambda \subset X$ of f is said to be* ***attracting of*** f *if it has a neighborhood U which is a trapping region such that*

$$\Lambda = \bigcap_{n \geq 0} f^n\left(\overline{U}\right).$$

We call U an ***isolating neighborhood of*** Λ*. Also, we call a set a* ***repelling set of*** f *is an attracting set of f^{-1}.*

Notice that, if Λ is an attracting set of f with isolating neighborhood U, then $U^* = X \backslash \overline{U}$ is a trapping region of f^{-1}. Indeed,

$$f^{-1}(\overline{U^*}) = f^{-1}\left(\overline{X \setminus \overline{U}}\right) \subseteq X \setminus f^{-1}\left(U\right) \subseteq X \setminus \overline{U} = U^*,$$

since $f(\overline{U}) \subseteq U$ implies $\overline{U} \subseteq f^{-1}(U)$. This proves that $\Lambda^* = \bigcap_{n \geq 0} f^{-n}(\overline{U^*})$ is a repelling set of f.

Definition 6 (Dual repelling)**.** *If Λ is an attracting set of f with isolating neighborhood U, then repelling set*

$$\Lambda^* = \bigcap_{n \geq 0} f^{-n}(\overline{U^*}),$$

with $U^* = X \setminus \overline{U}$, is called the ***dual repelling set of*** Λ.

Remark 1. It is worth noticing that both Λ and Λ^* are independent of the choice of isolating neighborhood U of Λ and that $f(\Lambda) = \Lambda$ and $f(\Lambda^*) = \Lambda^*$.

Lemma 1. *The set of attracting sets of f is countable.*

Proof. Since X is a compact metric space, there exists a countable basis of open sets $\mathcal{B}$ for X.

Let Λ be an attracting set of f with isolating neighborhood U. Since $\mathcal{B}$ is a basis, we can write $U = \bigcup_{i\in\mathbb{N}} U_i$, where each U_i belong to $\mathcal{B}$. By compactness of Λ, there is a finite number of those U_i, say $U_{i_1}, \ldots, U_{i_k}$, that cover Λ, i.e.,

$$\Lambda \subseteq U_{i_1} \cup \cdots \cup U_{i_k} \subseteq U.$$

Since U is isolating, we have:

$$\Lambda = \bigcap_{n\geq 0} f^n(U) = \bigcap_{n\geq 0} f^n(U_{i_1} \cup \cdots \cup U_{i_k}).$$

This shows that any attracting set is the intersection of forward iterations of a finite union of open sets on $\mathcal{B}$. Hence, there will be (at most) as many attracting sets as finite subsets of $\mathcal{B}$. Thus, the set of all attracting sets is countable. $\square$

Lemma 2. *Let $\{\Lambda_i\}_{i=1}^{\infty}$ be the attracting sets of f and let $\{\Lambda_i^*\}_{i=1}^{\infty}$ be their dual repelling sets. Then,*

$$\mathcal{CR}(f) = \bigcap_{i=1}^{\infty} (\Lambda_i \cup \Lambda_i^*)\,.$$

Proof. **First we prove the " $\subseteq$ " inclusion, i.e., that $\mathcal{CR}(f) \subseteq \bigcap_{i=1}^{\infty} (\Lambda_i \cup \Lambda_i^*)$.** Notice that, by Lemma 1, this is equivalent as proving that, for every attracting set Λ of f,

$$\mathcal{CR}(f) \subseteq (\Lambda \cup \Lambda^*)\,.$$

To do so, we show that for a point $x \in X$, if $x \notin (\Lambda \cup \Lambda^*)$ then $x \notin \mathcal{CR}(f)$.

Let U be an isolating neighborhood of Λ. Since $x \notin (\Lambda \cup \Lambda^*)$ and since $U^* = X \setminus U$ is an isolating neighborhood of Λ^*, there must be an order $N \in \mathbb{N}$ such that $x \notin \left(f^N(U) \cup f^{-N}(U^*)\right)$. Moreover, by definition of U^* and from the fact that $f(\overline{U}) \subseteq U$, we obtain:

$$X \setminus f^{-N}(U^*) = f^{-N}(\overline{U}) = f^{-N-1}\left(f(\overline{U})\right) \subseteq f^{-N-1}(U).$$

Since $x \in X \setminus f^{-N}(U^*)$ we conclude that $x \in f^{-N-1}(U)$.

Now, let $M \in \mathbb{N}$ be the smallest positive integer such that $x \in f^{-M}(U)$. Then $x \in f^{-M}(U) \setminus f^{-M+1}(U)$. Set $V = f^{-M}(U)$, which is also an isolating neighborhood for Λ. Since $X \setminus f(V)$ and $\overline{f^2(V)}$ are compact disjoint sets, the number

$$\varepsilon_1 = \frac{1}{2} d\left(X \setminus f(V), \overline{f^2(V)}\right)$$

is greater than zero.

Since $x \in f^{-M}(U) = V$, $f(x) \in f(V)$ and since $f(V)$ is open, there exists some $\varepsilon_2 > 0$ such that $\overline{B(f(x), \varepsilon_2)} \subseteq f(V)$.

Finally, since $f\left(\overline{B(f(x),\varepsilon_2)}\right) \subseteq f^2(V)$ and since $f\left(\overline{B(f(x),\varepsilon_2)}\right)$ is closed (hence compact in X) and $f^2(V)$ is again open, there exists some $\varepsilon_3 > 0$ such that

$$B\left(f\left(\overline{B(f(x),\varepsilon_2)}\right),\varepsilon_3\right) \subseteq f^2(V).$$

Here, for a compact subset $C \subseteq X$ and number $r > 0$, the set $B(C,r)$ is defined as

$$B(C,r) = \bigcup_{x\in C} B(x,r).$$

To show that $x \notin \mathcal{CR}(f)$, set $\varepsilon < \min\{\varepsilon_1,\varepsilon_2,\varepsilon_3\}$. Then, there is no $\varepsilon-$chain from x to x. Indeed, let $x, x_1, \ldots, x_k$ be an $\varepsilon-$chain. If, $k = 2$, then the chain cannot end in x, i.e., we cannot have $x_k = x$: if $x_2 = x$, we would have $d(f(x),x_1) < \varepsilon$ and $d(f(x_1),x) < \varepsilon$.

However, from $d(f(x),x_1) < \varepsilon < \varepsilon_2$, $x_1 \in B(f(x),\varepsilon_2) \subseteq f(V)$ and then $x_1 \in f(V)$. By also asking that $d(f(x_1),x) < \varepsilon < \varepsilon_2$ we must have

$$x \in B(f(x_1),\varepsilon_3) \subseteq B\left(f\left(\overline{B(f(x),\varepsilon_2)}\right),\varepsilon_3\right) \subseteq f^2(V).$$

Since $\varepsilon < \varepsilon_1 = \frac{1}{2}d\left(X \setminus f(V), \overline{f^2(V)}\right)$ we would have $x \in f(V) = f^{-M+1}(U)$, a contradiction with the choice of the order M ($x \in f^{-M}(U) \setminus f^{-M+1}(U) = V \setminus f(V)$).

Finally, even if $k > 2$ we would still have the same problem, since $f(x_2) \in f^3(V) \subseteq f^2(V)$ and if $d(f(x_2),x) < \varepsilon_1$ then $x \in f(V)$. This shows that there is no $\varepsilon-$chain from x to x, i.e., $x \notin \mathcal{CR}(f)$.

To prove the other inclusion "$\supseteq$", i.e., to prove that $\mathcal{CR}(f) \supseteq \bigcap_{i=1}^{\infty}(\Lambda_i \cup \Lambda_i^*)$, we proceed by contradiction: suppose that there is some $x \in \bigcap_{i=1}^{\infty}(\Lambda_i \cup \Lambda_i^*)$ such that $x \notin \mathcal{CR}(f)$. Then there is an $\varepsilon_0 > 0$ such that $x \notin \Omega(x,\varepsilon_0) = \{y \in X \mid x \xrightarrow{\varepsilon_0} y\}$.

As a consequence of item *(iv)* of Proposition 2, $\Omega(x,\varepsilon_0)$ is a trapping region, so $f(x) \in \Omega(x,\varepsilon_0)$. Notice that the set $\Lambda = \bigcap_{n\geq 0} f\left(\overline{\Omega(x,\varepsilon_0)}\right)$ has $\Omega(x,\varepsilon_0)$ as its isolating neighborhood.

Also, $x \notin \Lambda^*$, where Λ^* is the dual repelling set for Λ, since $\omega(x) \in \Omega(x,\varepsilon_0)$, where $\omega(x) = \overline{\{f^n(x) \mid n \geq 0\}}$ is the $\omega-$limit set of x by f.

Then $x \notin (\Lambda \cup \Lambda^*)$, contradicting the hypothesis that $x \in \bigcap_{i=1}^{\infty}(\Lambda_i \cup \Lambda_i^*)$. Hence, $x \in \mathcal{CR}(f)$.

□

This Lemma 2 states that the study of the chain recurrent is the same as the study of each pair of attracting-repelling sets.

Lemma 3. *If $x, y \in \mathcal{CR}(f)$, then $x \sim y$ if and only if, for every i, x and y are either both in Λ_i or both in Λ_i^*.*

Proof. Let $x, y \in \mathcal{CR}(f)$ and suppose that $x \sim y$. Let Λ be an attracting set of f. If $x \notin \Lambda$ and $y \notin \Lambda$, then x and y are both in Λ^* and we are done.

Suppose that at least some of them is in Λ, say $x \in \Lambda$. We need to prove that $y \in \Lambda$ as well. To do so, let U be an isolating neighborhood of Λ. Since $\overline{f(U)}$ and $X \setminus U$ are closed disjoint sets, there is a positive $\varepsilon > 0$ such that

$$0 < \varepsilon < d(X \setminus U, \overline{f(U)}).$$

From the same argument we have done to prove Lemma 2, we know that cannot exist an $\varepsilon/2-$chain (with length greater than 1) from any point of $f(U)$ to any point in $X \setminus U$. Hence, there is no $\varepsilon/2-$chain from a point in Λ to any point in Λ^*. Since $x \sim y$, this proves that y is also in Λ.

Reciprocally, suppose that x and y are **not** in the same class, i.e., $x \nsim y$. Then there is some ε_0 such that either $y \notin \Omega(x, \varepsilon_0)$ or $x \notin \Omega(y, \varepsilon_0)$. Suppose, without lost of generality, that $y \notin \Omega(x, \varepsilon_0)$. Let Λ be the attracting set with isolating neighborhood $\Omega(x, \varepsilon_0)$ and Λ^* be its dual repelling set. By Lemma 2, x and y are in $\Lambda \cup \Lambda^*$. Then, we must have $x \in \Lambda$ and $y \in \Lambda^*$. □

4 From trapping regions to Lyapunov functions

The dynamics outside of an attracting-repelling pair (Λ, Λ^*) is very simple since all points outside $\Lambda \cup \Lambda^*$ will move towards the attracting set Λ under forward iteration by f. This gives another step in our approach to mimic the dynamics of a source-sink map. Now, it remains to create a gradient-like map.

Definition 7 (Complete Lyapunov function)**.** *Given an homeomorphism $f\colon X \to X$ on a compact metric space (X, d), a continuous function $\Phi\colon X \to \mathbb{R}$ is called a **complete Lyapunov function for** f if it satisfies:*

(*i*) *for all $x \notin \mathcal{CR}(f)$, $\Phi(f(x)) < \Phi(x)$;*

(*ii*) *for every $x, y \in \mathcal{CR}(f)$, $\Phi(x) = \Phi(y)$ if and only if $x \sim y$;*

(*iii*) *$\Phi(\mathcal{CR}(f))$ is a compact nowhere dense subset of $\mathbb{R}$.*

*By analogy with the smooth setting, elements of $\Phi(\mathcal{CR}(f))$ are called **critical values** of Φ.*

Remark 2. Notice that item (*ii*) above implies that if $x \in \mathcal{CR}(f)$ then $\Phi(f(x)) = \Phi(x)$.

Lemma 4. *Let (Λ, Λ^*) be an attracting-repelling pair of f. There is a continuous function $\varphi\colon X \to [0,1]$ such that:*

(*i*) *$\varphi\big|_{\Lambda^*} = 1$;*

(*ii*) *$\varphi\big|_{\Lambda} = 0$;*

(*iii*) *for every $x \notin \Lambda \cup \Lambda^*$ we have $\varphi(x) \in (0,1)$ and $\varphi(f(x)) < \varphi(x)$.*

Proof. Consider the attracting-repelling pair (Λ, Λ^*), and $\psi\colon X \to [0,1]$ be the function defined by:

$$\psi(x) = \frac{d(x, \Lambda)}{d(x, \Lambda^*) + d(x, \Lambda)}.$$

Since Λ and Λ^* are compact disjoint sets, ψ is well-defined, continuous and satisfy $\psi(\Lambda^*) = 1$, $\psi(\Lambda) = 0$, and $\psi(x) \in (0,1)$ for all $x \notin \Lambda \cup \Lambda^*$.

Notice, however, that ψ does not give any sort of information of orbits of f. Then we define $\Psi(x)$ to be $\Psi(x) = \sup\{\psi(f^n(x)) \mid n \geq 0\}$. Then, automatically,

$$\Psi(f(x)) \leq \Psi(x),$$

for all $x \in X$. Moreover, since no point accumulates on Λ^* under positive iterations, we still get $\Psi^{-1}(0) = \Lambda$ and $\Psi^{-1}(1) = \Lambda^*$. We are going to show that Ψ is continuous.

Fix $x \in X$ and a sequence $(x_i)_i$ with $x_i \to x$. If $x \in \Lambda^*$, then $\Psi(x) = 1$ and since $\psi(x_i) \leq \Psi(x_i) \leq 1$. Since $\psi(x_i) \to 1$ by continuity of ψ, we must have $\Psi(x_i) \to 1$ as $i \to \infty$, i.e., Ψ is continuous for all points in Λ^*.

Now suppose $x \in \Lambda$. We need to show that $\Psi(x_i) \to 0 = \Psi(x)$. In order to do that, let U be an isolating neighborhood for Λ. Then, for every $\varepsilon > 0$ there is an order $N \in \mathbb{N}$ such that $f^N(U) \subseteq B(\Lambda, \varepsilon)$. So, if some $x_i \in f^N(U)$, all its forward iterates $f^n(x_i)$ are in $B(\Lambda, \varepsilon)$. Hence, $d(f^n(x_i), \Lambda^*) > K > 0$ for some positive constant K and $d(f^n(x_i), \Lambda^*) < \varepsilon$, both for all $n \geq 0$. Then, $\psi(f^n(x_i)) \leq \frac{\varepsilon}{K}$ for all $n \in \mathbb{N}$ and, therefore, $\Psi(x_i) = \sup\{\psi(f^n(x_i)) \mid n \geq 0\} \leq \frac{\varepsilon}{K}$. Since $x_i \to x$ and since $f^N(U)$ is a neighborhood of Λ, there is some $M \in \mathbb{N}$ such that $x_m \in f^N(U)$ for all $m \geq M$, proving that $\Psi(x_i) \to 0$, proving Ψ continuous.

Finally, we prove continuity in $X \setminus (\Lambda \cup \Lambda^*)$. Let $C = \overline{U} \setminus f(U)$, where U is still an isolating neighborhood of Λ and define $r = \inf\{\psi(x) \mid x \in C\}$. Since $f^n(C) \subseteq f^n(\overline{U})$ and $\Lambda = \bigcap_{n \geq 0} f^n(\overline{U})$, there is $n_0 \in \mathbb{N}$ such that $\psi(f^n(C)) \subseteq [0, r/2]$ for all $n > n_0$. So, for $x \in C$,

$$\Psi(x) = \sup\{\psi(f^n(x)) \mid n \geq 0\} = \max\{\psi(f^n(x)) \mid 0 \leq n \leq n_0\}.$$

Hence, $\Psi\big|_C = \max\{\psi \circ f^n \mid 0 \leq n \leq n_0\}$, which is continuous. But the "bands" C form a partition for $X \setminus (\Lambda \cup \Lambda^*)$ since $\bigcup_{n \in Z} f^n(C) = X \setminus (\Lambda \cup \Lambda^*)$. Thus we have proven that Ψ is continuous in X.

Now, define $\varphi \colon X \to [0, 1]$ by

$$\varphi(x) = \sum_{n=0}^{\infty} \frac{\Psi(f^n(x))}{2^{n+1}}.$$

Since Ψ is continuous on a compact space, it is bounded, so φ is well-defined and also continuous. Moreover, it satisfy $\varphi^{-1}(0) = \Lambda$ and $\varphi^{-1}(1) = \Lambda^*$, because Ψ satisfies it.

If $x \notin (\Lambda \cup \Lambda^*)$ then φ is strictly decreasing along orbits of x. To see that, observe that

$$\varphi(f(x)) - \varphi(x) = \sum_{n=0}^{\infty} \frac{\Psi(f^{n+1}(x)) - \Psi(f^n(x))}{2^{n+1}},$$

hence $\varphi(f(x)) - \varphi(x) = 0$ if and only if $\Psi(f^{n+1}(x)) - \Psi(f^n(x)) = 0$ for all n, since Ψ is non-increasing along orbits. So, $\varphi(f(x)) - \varphi(x) = 0$ if and only if Ψ is constant along orbits of f. However, this cannot happen because, Ψ is continuous and since, $d(f^n(x), \Lambda) \to 0$, there is some subsequence of $(f^n(x))_n$ converging to some point in Λ. If it were constant along orbits, it would be constant equal to zero, implying $x \in \Lambda$, a contradiction. This proves $\varphi(f(x)) < \varphi(x)$ for all $x \notin (\Lambda \cup \Lambda^*)$, and hence the Lemma. $\square$

5 Conley's Theorem

Theorem (Conley)**.** *Every homeomorphism $f\colon X \to X$ on a compact metric space X admits a complete Lyapunov function.*

Proof. From Lemma 1 we know that there are countable many attracting sets, say $\{\Lambda_i\}_{i\in\mathbb{N}}$. For each Λ_i, Lemma 4 above gives us a function $\varphi_i\colon X \to [0,1]$ which is 0 on Λ_i, 1 on Λ_i^*, and is strictly decreasing on orbits of points $x \notin (\Lambda \cup \Lambda^*)$.

Define a function $\Phi\colon X \to \mathbb{R}$ by

$$\Phi(x) = 2 \cdot \sum_{i=1}^{\infty} \frac{\varphi_i(x)}{3^i}.$$

We are going to show that Φ is a complete Lyapunov function for f.

Since each φ_i is continuous and bounded between 0 and 1, the series converges uniformly. This shows that Φ is continuous.

From Lemma 2, we know that all $x \in \mathcal{CR}(f)$ belongs to $\Lambda_i \cup \Lambda_i^*$ for all $i \in \mathbb{N}$, i.e., for each $i \in \mathbb{N}$, $\varphi_i(x) = 0$ or $\varphi_i(x) = 1$. So, if $x \in \mathcal{CR}(f)$,

$$\Phi(x) = 2 \cdot \sum_{n=1}^{\infty} \frac{\varphi_i(x)}{3^i} = \sum_{n=1}^{\infty} \frac{a_i}{3^i},$$

where $a_i \in \{0, 2\}$. This is equivalent to saying that the expression of the real number $\Phi(x)$ on base 3, only contains digits 0 and 2. This shows that $\Phi\left(\mathcal{CR}(f)\right) \subset \mathscr{C}$, where $\mathscr{C}$ is the usual middle-third Cantor set in $[0,1]$. Hence, $\Phi\left(\mathcal{CR}(f)\right)$ is a compact nowhere dense set in $\mathbb{R}$ and Φ satisfy condition (iii) at the definition of Lyapunov function.

If $x \notin \mathcal{CR}(f)$, then Lemma 1 implies that there is some $n \in \mathbb{N}$ such that $x \notin \Lambda_n \cup \Lambda_n^*$. So, $\varphi_n(f(x)) < \varphi_n(x)$. Since $\varphi_i(f(x)) \leq \varphi_i(x)$ for all $i \in \mathbb{N}$, this implies $\Phi(f(x)) < \Phi(x)$ and proves Φ satisfy condition (i) on the definition.

Finally, we need to show that, for any $x, y \in \mathcal{CR}(f)$, $\Phi(x) = \Phi(y)$ if and only if $x \sim y$. To do so we use Lemma 3. To say that $\Phi(x) = \Phi(y)$ is the same as saying that $\Phi(x)$ and $\Phi(y)$ have the same expansion on base 3 on the middle-third Cantor set $\mathscr{C}$. This is equivalent as saying that $2\varphi_i(x) = 2\varphi_i(y) \in \{0, 2\}$ for all $i \in \mathbb{N}$, i.e., there is no $i \in \mathbb{N}$ for which some of the x and y is in Λ_i while the other is in Λ_i^*. Hence, Lemma 3 guarantees that this is equivalent to saying that $x \sim y$, showing that Φ satisfies condition (ii). $\square$

References

[1] John Franks. Notes on Chain Recurrence and Lyapunonv Functions. *arXiv preprint arXiv:1704.07264*, 2017.

[2] Douglas E Norton. The fundamental theorem of dynamical systems. *Commentationes Mathematicae Universitatis Carolinae*, 36(3):585–597, 1995.

[3] Lan Wen. *Differentiable Dynamical Systems*, volume 173. American Mathematical Soc., 2016.

[4] Conan Wu. Fundemental Theorem of Dynamical Systems (Part 2), 2010.