Notes about *Fubini Foiled*

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1 Introduction

These notes are based on some study notes I took during the program Big Ideas in Dynamics to better understand the paper *Fubini foiled: Katok's paradoxical example in measure theory*, by John Milnor [7]. In this paper, Milnor explains the construction of an example of a foliation $\mathcal{F}$ of the square $(0,1) \times [0,1]$ and of a measurable set E on the same square, such that: the Lebesgue measure of E is 1, but each leaf of $\mathcal{F}$ intersects E in at most one point. As we are going to point out, this implies that the foliation $\mathcal{F}$ fails to satisfy the property of absolute continuity.

The main goal of these notes is to help someone who is trying to understand these topics for the first time. Therefore, I believe it is important to point out some references I followed: of course, the original paper [7] by Milnor, which is extremely well-written; next, and I believe this is a good reference for several topics in Dynamical Systems and Ergodic Theory, we followed the post *Fubini Foiled* on Vaugh Climenhaga's Math Blog; finally, for those who can read math in Portuguese, the master's dissertation [8] of Marcielis Espitia Noriega from UNICAMP, named *Ergodicity of Anosov diffeomorphisms of class* C^2. There, she not only explains the construction we are dealing with but also makes a good introduction to Hyperbolic Dynamics and proves Hopf's Argument.

It is also very important to thank Davi Obata for his patience and explanations during office hours and via emails, Kate Holmes, Homin Lee, and Zihan Xia for studying together during the program, and finally Benjamin Call and Noelle Sawyer for organizing this very nice program!

One last thing before we begin with the math: this text may contain errors! If someone finds anything, I would love to be contacted so that I can correct it. I can be reached at odylocosta@gmail.com.

2 A brief introduction to foliations

In this Section, we give a brief introduction to foliations. It is a piece of Chapter 2 of my master's dissertation [3], named *Unique ergodicity of the horocycle flow via hyperbolic dynamics* and we follow mainly [1].

Throughout the Section, (M, g) will be closed (compact without boundary) and connected Riemannian manifold.

2.1 Flows

A flow is a map $\varphi\colon \mathbb{R} \times M \to M$ satisfying:

- $\varphi(0, x) = x$ for all $x \in M$;
- $\varphi(s, \varphi(t, x)) = \varphi(s + t, x)$ for all $x \in M$ and $s, t \in \mathbb{R}$.

Unless we explicitly say otherwise, we will always assume the flow φ to be of class C^r, with $r \geq 1$. In particular, for each $t \in \mathbb{R}$, the map $\varphi_t\colon M \to M$ defined by $\varphi_t(x) = \varphi(t, x)$, always is of class C^r.

The existence of flows on manifolds is intimately related to the existence of vector fields, as the next example tells us.

Example 1 (Flows and vector fields)**.** *Consider a vector field $X \in \mathfrak{X}^r(M)$. Then, the Fundamental Theorem of ODE's guarantees that, through each point $p \in M$, the initial value problem*

$$\begin{cases} x'(t) = X(x(t)) \\ x(0) = p \end{cases}$$

admits a unique solution $\gamma_p\colon \mathbb{R} \to M$. Moreover, the map $\varphi\colon \mathbb{R} \times M \to M$ defined by $(t, p) \mapsto \gamma_p(t)$ is a flow of class C^r such that

$$\frac{\partial \varphi(t, p)}{\partial t} = X(\varphi(t, p)).$$

Reciprocally, to each flow φ_t on a manifold M, there is a vector field X that it integrates: one just has to define $X(p) = X(\varphi_0(p)) = \frac{\partial \varphi(0,p)}{\partial t}$, for each $p \in M$.

A general statement, as well as a proof, of the Fundamental Theorem of ODE's can be found in [6], as Theorem 9.12, p. 212. Also, the fact that flows on compact manifolds are complete, i.e., are well-defined over $\mathbb{R}$, is proved in the same text: see Corollary 9.17, p. 216.

We now give a concrete example of a flow on a compact manifold. To do so, we explicit the construction of the quotient $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$. Define, on $\mathbb{R}^n$, the following equivalence relation: we say two points $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ in $\mathbb{R}^n$ are equivalent if and only if their difference is an integer vector. More explicitly,

$$(x_1, \ldots, x_n) \sim (y_1, \ldots, y_n) \iff (x_1 - y_1, \ldots, x_n - y_n) \in \mathbb{Z}^n.$$

We denote by $[x]$ or $[(x_1, \ldots, x_n)]$ the equivalence class of the point $x = (x_1, \ldots, x_n)$. Finally, we define $\mathbb{T}^d$ to be the quotient of $\mathbb{R}^n$ by this equivalence relation: $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$.

Note that $\mathbb{T}^n$ is an abelian group with the operation:

$$[(x_1, \ldots, x_n)] + [(y_1, \ldots, y_n)] = [(x_1 + y_1, \ldots, x_n + y_n)].$$

Example 2 (Linear flow on $\mathbb{T}^n$)**.** *Let $\theta = (\theta_1, \ldots, \theta_n) \in \mathbb{R}^n$ be a fixed vector and let $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ be the $n-$dimensional torus endowed with the volume measure μ. Define the* ***linear flow*** *φ_t* ***on*** *$\mathbb{T}^n$* ***in the direction of*** *θ as the map $\varphi_t \colon \mathbb{T}^n \to \mathbb{T}^n$ such that to each $[x] = [(x_1, \ldots, x_n)] \in \mathbb{T}^n$ associates*

$$\varphi_t(x) = [x + t\theta].$$

In this example, the linear flow φ_t is the solution of the following ODE on $\mathbb{T}^n$:

$$\frac{dx}{dt} = \theta.$$

As a general goal in Dynamical Systems, given a flow φ on M, we want to know what happens to its orbits:

$$\mathcal{O}_\varphi(x) \coloneqq \{\varphi(t, x) \in M \mid t \in \mathbb{R}\},$$

for each $x \in M$.

Since we are mainly dealing with invertible systems, it makes sense to break the orbit of each point $x \in M$ into two subsets: the ***positive semi-orbit*** and the ***negative semi-orbit*** by the flow φ. Respectively, they are defined as follows:

- $\mathcal{O}^+_\varphi(x) \coloneqq \{\varphi(t, x) \in M \mid t \geq 0\}$;

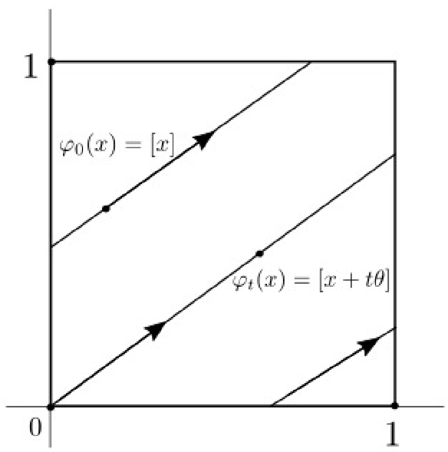

Figure 1: The flow $\varphi_t(x) = [x + t\theta]$ on $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$.

- $\mathcal{O}^-_\varphi(x) := \{\varphi(t,x) \in M \mid t \leq 0\}$.

With these definitions, $\mathcal{O}_\varphi(x) = \mathcal{O}^-_\varphi(x) \cup \mathcal{O}^+_\varphi(x)$.

If, for a point $p \in M$ there exists a time $T \in \mathbb{R}$ such that $\varphi_T(p) = p$, we call the point p a ***periodic point*** for φ. Also in this setting, and we say the orbit of p is ***closed*** and if $\tau \in \mathbb{R}$ is such that $\varphi_\tau(p) = p$ and for all $0 < t < \tau$, we have $\varphi_t(p) \neq p$, then we say the orbit of p closed of period τ. The set of all periodic points p for φ is denoted by $\mathrm{Per}\,(\varphi)$.

In general, flows can have plenty, few, or even none periodic orbits. Even in the simple setting of Example 2, the orbits of point through the flow behave very differently depending on the direction vector θ:

Proposition 1. *Consider $\theta = (\theta_1, \ldots, \theta_n) \in \mathbb{R}^n$. If each θ_i is rational, say $\theta_i = \frac{p_i}{q_i}$ with $p_i, q_i \in \mathbb{Z}$, $q_i \neq 0$ and $\gcd(p_i, q_i) = 1$ for each $i = 1, \ldots, n$, then each point $x \in \mathbb{T}^n$ is periodic.*

Proof. Indeed, consider $T = \mathrm{lcm}\,(q_1, \ldots, q_n)$. Then,

$$\varphi_T(x) = [x + T\theta] = [x],$$

for all $x \in \mathbb{T}^n$. $\square$

On the opposite direction of the above proposition, if α is an irrational number, then the linear flow in the direction of $\theta = (\alpha, 0, \ldots, 0)$ has no periodic points: for each $x \in \mathbb{T}^n$, the orbit $\varphi_t(x)$ remains in a vertical circle, on which the dynamics is an irrational rotation by α. Hence, this linear flow has no periodic orbit. In fact, it can be shown that if the coordinates of the direction vector $\theta = (\theta_1, \ldots, \theta_n) \in \mathbb{R}^n$ are rationally independent, i.e., if for every $k \in \mathbb{Z}^n$ such that $\langle k, \theta \rangle = 0$ we have that $k = (0, \ldots, 0)$, then the flow φ_t is **minimal**: each of its orbits is dense on $\mathbb{T}^n$.

2.2 Foliations

This subsection has as its main objective to define foliations and to present their relation to flows with certain regularity. In particular, we want to stress the fact that the orbits of the linear flow on $\mathbb{T}^n$ produce a foliation with several dynamical properties of great interest.

Definition 1 (Foliation). *Let M be a smooth manifold of dimension m. A C^r foliation of dimension n in M is a C^r atlas $\mathcal{F}$ on M which is maximal with the following properties:*

(a) *If (U, φ) is a chart in $\mathcal{F}$, then $\varphi(U) = U_1 \times U_2 \subset \mathbb{R}^n \times \mathbb{R}^{m-n}$ where U_1 and U_2 are open discs in $\mathbb{R}^n$ and $\mathbb{R}^{m-n}$, respectively;*

(b) *If (U, φ) and (V, ψ) are charts in $\mathcal{F}$ such that $U \cap V \neq \emptyset$ then the change of coordinates map $\psi \circ \varphi^{-1} \colon \varphi(U \cap V) \to \psi(U \cap V)$ is of the form*

$$\psi \circ \varphi^{-1}(x, y) = (h_1(x, y), h_2(y)),$$

where h_1 and h_2 are C^r diffeomorphisms with $(x, y) \in (U_1 \cap V_1) \times (U_2 \cap V_2)$.

Whenever M admits such an atlas $\mathcal{F}$, we say that M is foliated by $\mathcal{F}$, or that $\mathcal{F}$ is a foliated structure of dimension n and class C^r on M, and call the charts $(U, \varphi) \in \mathcal{F}$ foliation charts.

Example 3. *Our first example of foliation is the example o a foliation defined by a submersion.*

Let $f\colon M \to N$, a C^r submersion between manifolds M and N of dimensions m and n, respectively. Given a point $p \in M$ we can use the local form of the submersions to obtain local charts (U, φ) on M and (V, ψ) on N, such that $p \in U$, $f(p) \in V$, $\varphi(U) = U_1 \times U_2 \subseteq \mathbb{R}^{m-n} \times \mathbb{R}^n$, and $\psi(V) = V_2 \supset U_2$ and the composition $\psi \circ f \circ \varphi^{-1}\colon U_1 \times U_2 \to U_2$ has the form of a projection π_2 to second coordinate in $\mathbb{R}^m = \mathbb{R}^{m-n} \times \mathbb{R}^n$: $\psi \circ f \circ \varphi^{-1}(x, y) = y$, as shown in Figure 2 below.

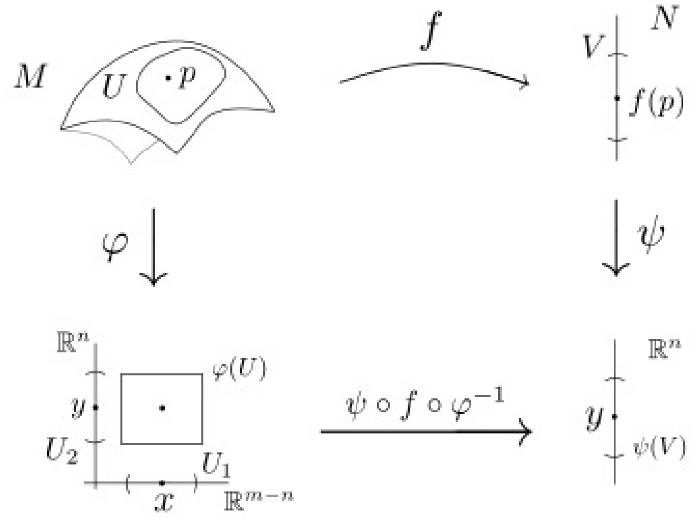

Figure 2: Local form of the submersions.

From that we obtain a C^r−foliation $\mathcal{F}$ of dimension n on M: for the foliated charts we choose, for each point $p \in M$, the chart (U, φ) which satisfies the local form of the submersions with some local chart (V, ψ) over $f(p)$.

To check that $\mathcal{F}$ is indeed a foliation we only need to see the condition of compatibility of the charts: let (U, φ) and $(\widetilde{U}, \widetilde{\varphi})$ be charts in $\mathcal{F}$ such that $U \cap V \neq \emptyset$. So we must prove that, on $\varphi(U \cap \widetilde{U})$, one can write:

$$\widetilde{\varphi} \circ \varphi^{-1}(x, y) = (h_1(x, y), h_2(y)).$$

In order to do that, pick $p \in U \cap \widetilde{U}$ and let (V, ψ) and $(\widetilde{V}, \widetilde{\psi})$ be charts on N over $f(p)$ such that, on $\varphi(U \cap \widetilde{U})$ and on $\widetilde{\varphi}(U \cap \widetilde{U})$ we have:

$$\psi \circ f \circ \varphi^{-1}\big|_{\varphi(U)} = \pi_2\big|_{\varphi(U)} \tag{1}$$

and

$$\widetilde{\psi} \circ f \circ (\widetilde{\varphi})^{-1}\Big|_{\widetilde{\varphi}(\widetilde{U})} = \pi_2\big|_{\widetilde{\varphi}(\widetilde{U})}. \tag{2}$$

Therefore, if we write $\widetilde{\varphi} \circ \varphi^{-1}\colon \varphi\left(U \cap \widetilde{U}\right) \to \widetilde{\varphi}\left(U \cap \widetilde{U}\right)$ as

$$\widetilde{\varphi} \circ \varphi^{-1}(x, y) = (h_1(x, y), h_2(x, y)),$$

we have:

$$\begin{aligned}
h_2(x,y) &= \pi_2 \circ \widetilde{\varphi} \circ \varphi^{-1}(x,y) \\
&= \widetilde{\psi} \circ f \circ \widetilde{\varphi}^{-1} \circ \widetilde{\varphi} \circ \varphi^{-1}(x,y) && \text{(by 2)} \\
&= \widetilde{\psi} \circ f \circ \varphi^{-1}(x,y) \\
&= \widetilde{\psi} \circ \psi^{-1} \circ \psi \circ f \circ \varphi^{-1}(x,y) \\
&= \widetilde{\psi} \circ \psi^{-1} \circ \pi_2(x,y) && \text{(by 1)} \\
&= \widetilde{\psi} \circ \psi^{-1}(y),
\end{aligned}$$

meaning that we can write $h_2(x,y)$ simply as $h_2(y)$, as we wished. Hence, (U,φ) is a foliated chart of the foliated structure $\mathcal{F}$ of dimension n and class C^r on M.

Definition 2. *Given a C^r foliation $\mathcal{F}$ of dimension n on a $m-$dimensional smooth manifold M (where $0 < n < m$). Consider a local chart $(U,\varphi) \in \mathcal{F}$ such that $\varphi(U) = U_1 \times U_2 \subseteq \mathbb{R}^n \times \mathbb{R}^{m-n}$. We call the sets of the form $\varphi^{-1}(U_1 \times \{c\})$, with $c \in U_2$, the **plaques of** U (or of $\mathcal{F}$).*

*A **path of plaques of** $\mathcal{F}$ is a sequence $\alpha_1, \ldots, \alpha_k$ of plaques of $\mathcal{F}$ such that $\alpha_j \cap \alpha_{j+1} \neq \emptyset$ for all $j \in \{1, \ldots, k-1\}$. Moreover, since we can cover M by plaques of $\mathcal{F}$, we can define the following equivalence relation on M:*

$$p \sim q \iff \text{there exists a path of plaques } \alpha_1, \ldots, \alpha_k \text{ with } p \in \alpha_1 \text{ and } q \in \alpha_k.$$

*The equivalence classes of the relation $\sim$ on M are called **leaves** of the foliation $\mathcal{F}$.*

Notice that, given a local chart $(U,\varphi) \in \mathcal{F}$ such that $\varphi(U) = U_1 \times U_2 \subseteq \mathbb{R}^n \times \mathbb{R}^{m-n}$ as above, if we fix a point $c \in U_2$, the map $\varphi^{-1}\big|_{U_1 \times \{c\}} : U_1 \times \{c\} \to U$ is a C^r embedding. Remembering that U_1 is a open disc, the plaques are path-connected $n-$dimensional C^r submanifolds of M.

Therefore, if p and q in M are in the same leaf of $\mathcal{F}$, there is a path of plaques connecting the two and, moreover, there is a continuous path connecting them because $\alpha_j \cap \alpha_{j+1} \neq \emptyset$ for all $j \in \{1, \ldots, k-1\}$ and the plaques are path-connected.

Example 4. *In Example 3 the leaves are the connected components of the level sets $f^{-1}(c)$, where $c \in N$.*

Example 5. *Let $f : \mathbb{R}^3 \to \mathbb{R}$ be a submersion defined by*

$$f(x,y,z) = \alpha(x^2 + y^2) \cdot e^z,$$

where $\alpha : \mathbb{R} \to \mathbb{R}$ is a C^∞ function such that $\alpha(1) = 0$, $\alpha(0) = 1$ and if $t > 0$ then $\alpha' < 0$.

Using the construction of the Example 3, let $\mathcal{F}$ be the foliation of $\mathbb{R}^3$ whose leaves are the connected components of the submanifolds $f^{-1}(c)$, for $c \in \mathbb{R}$.

The leaves of $\mathcal{F}$ are of three types, all ruled by the relation with the solid cylinder

$$C = \{(x,y,z) \in \mathbb{R}^3 \mid x^2 + y^2 \leq 1\},$$

in the following way:

(i) *the boundary of C, i.e., $\partial C = \{(x,y,z) \in \mathbb{R}^3 \mid x^2 + y^2 = 1\}$ is a leaf of $\mathcal{F}$;*

(*ii*) *outside* C*, i.e., on the set of points* $(x,y,z) \in \mathbb{R}^3$ *such that* $x^2+y^2>1$*, the leafs of* $\mathcal{F}$ *are all homeomorphic to cylinders;*

(*iii*) *finally, in the interior of* C*, i.e., on the set of points* $(x,y,z) \in \mathbb{R}^3$ *such that* $x^2+y^2<1$*, the leafs of* $\mathcal{F}$ *are all homeomorphic to* $\mathbb{R}^2$ *by a parametrization* $\sigma\colon \mathbb{D}^2 \to \mathbb{R}^3$ *from the disk* $\mathbb{D}^2 = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2<1\}$ *to* $\mathbb{R}^3$*, defined by:*

$$\sigma(x,y) = \left(x, y, \log\left(\frac{c}{\alpha(x^2+y^2)}\right)\right).$$

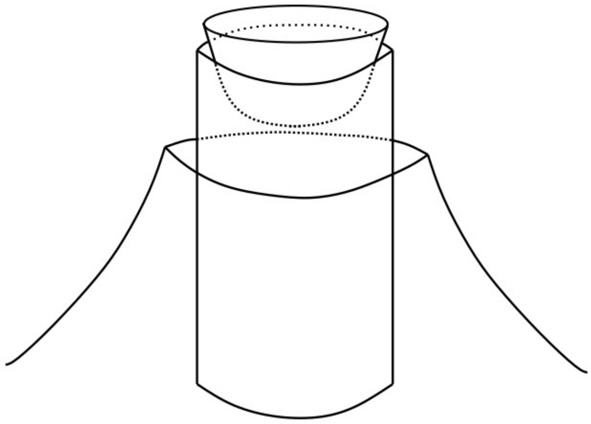

Figure 3: Example of foliation coming from a submersion.

The next example is the main example of this Section and has a very dynamical nature.

Example 6. *Foliations arising from vector fields without singularities.*

Let X be a C^r $(r \geq 1)$ vector field without singularities on a compact manifold M (with $\dim M = m$). As we have seen in Example 1, associated to X we have a flow $\varphi(t,x)$ such that

$$X(\varphi(t,x)) = \frac{\partial \varphi(t,x)}{\partial t}$$

for every $(t,x) \in \mathbb{R} \times M$.

Let $i\colon \mathbb{B}^{m-1}(0) \to M$ be an embedding of a small $m-1$ disk around $0 \in \mathbb{R}^m$, such that $i(0)=p$, that is transverse to X everywhere. Since $X(p) \neq 0$, for $\varepsilon > 0$, the map

$$\Phi\colon \mathbb{B}^{m-1}(0) \times (-\varepsilon, \varepsilon) \to M$$

defined by

$$\Phi(x,t) = \varphi(t, i(x))$$

has maximal rank at $(0,0) \in \mathbb{B}^{m-1}(0) \times (-\varepsilon,\varepsilon)$.

By the Inverse Mapping Theorem, there is a neighborhood $V \subset M$ around p such that $\Phi^{-1}\big|_V$ is a diffeomorphism between V and a product neighborhood $\tilde{\mathbb{B}}^{m-1}(0) \times (-\varepsilon', \varepsilon') \subseteq \mathbb{B}^{m-1}(0) \times (-\varepsilon,\varepsilon)$ of $(0,0)$. This is a local chart for the one-dimensional foliation on M defined by the curves $t \mapsto \varphi_t(x)$, the integral curves of X.

Therefore, from a regular vector X on M, we obtain a one-dimensional foliation $\mathcal{F}$ whose leafs are the integral curves of X.

Example 7 (Linear flow on $\mathbb{T}^n$)**.** *A particular case of the previous example is the case where $M = \mathbb{T}^n$ and $X(x) = \theta$ for all $x \in \mathbb{T}^n$, where $\theta \in \mathbb{R}^n$.*

From Example 2, we know that the solutions of the ODE $\frac{dx}{dt} = X(x)$ is $\varphi_t(x) = [x + t\theta]$. Hence, in the particular case of the foliation obtained from X, the leafs are

$$L_x = \{\varphi_t(x) \mid t \in \mathbb{R}\}.$$

Now, we briefly comment on how we could extend the relation between foliations on manifolds M and higher dimensional analogs of vector fields.

Definition 3. *A field of $k-$planes on a manifold M is a map $P\colon M \to G_k(TM)$*[1] *which associates each point $x \in M$ a $k-$dimensional vector subspace $P(x) \subset T_xM$. In the particular case of $k = 1$, we call the map P a* ***line field****.*

We say a $k-$plane field P on M is of class C^r if, for every $q \in M$, there exist k vector fields $X_1, \ldots, X_k$ defined in a neighborhood V of q and of class C^r, and such that $\{X_1(x), \ldots, X_k(x)\}$ is a basis for $P(x)$ for every $x \in V$.

Definition 4. *Given a $k-$plane field P on M, we call a submanifold $N \subset M$ an* ***integral manifold of*** *P if $T_xN = P(x)$ for every $x \in N$. We say P is* ***integrable*** *if there exists a foliation $\mathcal{F}$ such that, for every point $x \in M$ there exists a leaf $\mathcal{F}(x)$ of $\mathcal{F}$ such that $T_x(\mathcal{F}(x)) = P(x)$. Moreover, we say that P is* ***uniquely integrable*** *if the foliation above is unique.*

Definition 5. *We say a plane field P is* ***completely integrable*** *if, given two vector fields X and Y such that, for each $q \in M$, if $X(q)$ and $Y(q)$ are in $P(q)$, then $[X, Y](q) \in P(q)$, where $[\cdot, \cdot]$ is the Lie bracket on M.*

Finally, we present a theorem of Frobenius that generalizes to plane fields the existence of tangent foliations:

Theorem. *Let P be a C^r $k-$plane field (for $k \geq 1$) on M. If P is completely integrable, then there exists a C^r foliation $\mathcal{F}$ of dimension k on M such that $T_q(\mathcal{F}) = P(q)$ for all $q \in M$. Conversely, if $\mathcal{F}$ is a C^r ($r \geq 2$) foliation and P is the tangent plane field to $\mathcal{F}$, then P is uniquely integrable.*

3 Fubini Foiled

As explained in the Introduction, we are going to present an example of a measurable set E on the square $(0, 1) \times [0, 1]$ and foliation of the same square such that the Lebesgue measure of E is 1 but each leaf of E only intersects E at a single point. In this Section, we will follow [2] closely.

To point out how weird this example seems at first sight, we recall Fubini's Theorem (following [9]):

Theorem 1 (Fubini)**.** *Let $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ be two measure spaces and suppose ν is a complete measure. Let $f\colon X \times Y \to \mathbb{R}$ be an integrable function with respect two the product measure $\mu \times \nu$. Then, for μ-almost every $x \in X$, the function $f(x, \cdot)\colon Y \to \mathbb{R}$ is integrable over Y with respect to ν and:*

$$\int_{X\times Y} f d(\mu \times \nu) = \int_X \left(\int_Y f(x, y) d\nu(y) \right) d\mu(x). \tag{3}$$

[1] Here $G_k(TM)$ denotes the Grassmannian manifold of $k-$dimensional subspaces of TM.

For the particular case of $\mathbb{R}^n$, we may state the theorem with some additional information as follows:

Theorem 2 (Fubini for $\mathbb{R}^n$)**.** *For natural numbers $n, m,$ and k, such that $n = m + k$, consider:*

- *μ_i to be the Lebesgue measure on $\mathbb{R}^i$, for $i = n, m, k$;*
- *$\varphi\colon \mathbb{R}^n \to \mathbb{R}^m \times \mathbb{R}^k$ the function defined by*

$$\varphi(x_1, \ldots, x_n) = ((x_1, \ldots, x_m), (x_{m+1}, \ldots, x_n)) \in \mathbb{R}^m \times \mathbb{R}^k,$$

for every $(x_1, \ldots, x_n) \in \mathbb{R}^n$.

Then, a function $f\colon \mathbb{R}^m \times \mathbb{R}^k \to \mathbb{R}$ is measurable with respect to the product measure $\mu_m \times \mu_k$ if, and only if, $f \circ \varphi$ is measurable with respect to the Lebesgue measure μ_n. Moreover, if f is integrable over $\mathbb{R}^n$ with respect to the Lebesgue measure μ_n, then

$$\int_{\mathbb{R}^n} f d\mu_n = \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} f(x, y) d\mu_m(x) \right) d\mu_k(y). \tag{4}$$

So, for example, consider the foliation $\mathcal{F}$ of the open unite square $I^2 = (0,1)^2$ by horizontal lines, i.e., each leaf L_α of $\mathcal{F}$ is $L_\alpha = (0,1) \times \{\alpha\}$ for $\alpha \in (0,1)$. If we call m the Lebesgue measure of U, and m_α the Lebesgue measure restricted to each leaf L_α, we would be able to apply Fubini to write, for every measurable set $E \subset U$, the following characterization of the measure m:

$$m(E) = \int_{I^2} \chi_E dm = \int_I \left(\int_{L_\alpha} \chi_E(z) dm_\alpha(z) \right) d\alpha,$$

where χ_E is the characteristic function of the set E.

Next, we could ask for more generality: suppose $\mathcal{F}$ is a foliation of the open unit square I^2 by curves L_α that we assume to be graph of smooth functions $\phi_\alpha\colon I \to I$, i.e.,

$$L_\alpha = \{(x, \phi_\alpha) \mid x \in I\}.$$

Can we do the same? If we ask for regularity, the answer is still **yes**.

More precisely, define the map $\Phi\colon I^2 \to I^2$ by $\Phi(x, y) = \phi_y(x)$. Clearly, if we fix y and vary x we get the leaf L_y of $\mathcal{F}$, so that the foliation $\mathcal{F}$ is the image of the foliation of I^2 by horizontal lines under the map Φ. If the map Φ depends smoothly on x and y, then we have a smooth foliation. Notice that this is not a vacuous assumption since, at this point, we only have asked that the leaves of the foliation are smooth, and hence that Φ depends smoothly on x. However, we haven't said anything about the dependence on y, i.e., on the transverse direction: Φ could not depend smoothly on it. But indeed, if we ask that Φ depends smoothly on x and y, then we have a smooth foliation, not only smooth leaves. As a consequence, we can write, by a variation of Fubini's Theorem on this setting, that the measure of any measurable set $E \subset I^2$ can be written as

$$m(E) = \int_I \left(\int_{L_\alpha} \rho_\alpha(z) \cdot \chi_E(z) dm_\alpha(z) \right) d\alpha,$$

for a $L^1(m_\alpha)$ density $\rho_\alpha\colon L_\alpha \to \mathbb{R}$ that can be determined in terms of the derivative of Φ.

Indeed, if we call J the Jacobian determinant of $\Phi(x,y)=(x',y')$, Fubini's Theorem tells us that

$$m(E)=\int_I\int_I \frac{1}{J(x,y)}\cdot\chi_E(x,y)dx'dy'.$$

Next, let J_α be the Jacobian derivative of $\Phi\big|_{W_\alpha}$ and hence $dx'=J_\alpha\cdot dm_\alpha$. Also, observe that $\Phi(W_\alpha)=(0,1)\times\{y(\alpha)\}$ where $\alpha\mapsto y(\alpha)$ is differentiable, and then $dy'=y'(\alpha)d\alpha$. So, if we set $\rho_\alpha=\frac{J_\alpha\cdot y'}{J}$, we get the desired formula.

So we now know that in the regular case, we can still apply Fubini in some sense: however, we had to ask on regularity not only in the x direction but also in the y direction in the above example. It is on this "gap" of non-smoothness dependence on the transverse direction that we construct the example.

3.1 Construction of the example

For each $p\in(0,1)$ consider the map $f_p\colon[0,1]\to[0,1]$ defined by:

$$f_p(x)=\begin{cases}\frac{x}{p} & \text{if } x\in I_0(p):=[0,p)\\ \frac{x-p}{1-p} & \text{if } x\in I_1(p):=[p,1]\end{cases}. \tag{5}$$

Observe that $f_p(0)=0$ and $f_p(1)=1$ for $p\in(0,1)$. So, if we make identify the closed interval with the circle $\mathbb{R}/\mathbb{Z}$, we can still define f_p, now from $\mathbb{R}/\mathbb{Z}$ to itself. We are going to be thinking about $\mathbb{R}/\mathbb{Z}$ from now on.

Claim. *The function f_p preserves the Lebesgue measure m on $\mathbb{R}/\mathbb{Z}$. Moreover, the pair (f_p,m) is measure-theoretic conjugated to the pair (σ,μ_p), where $\sigma\colon\Sigma\to\Sigma$ is the shift map over $\Sigma=\{0,1\}^{\mathbb{N}}$ and μ_p is the $(p,1-p)$-Bernoulli measure on Σ.*

Proof. To see that f_p, consider a measurable set $J\subset\mathbb{R}/\mathbb{Z}$. Then, since $f_p^{-1}(J)=J_0\cup J_1$ is the union of two disjoint intervals $J_0\subset I_0(p)$ of length $p\cdot m(J)$ and $J_1\subset I_1(p)$ of length $(1-p)\cdot m(J)$, we conclude:

$$m(f_p^{-1}(J))=m(J_0\cup J_1)=p\cdot m(J)+(1-p)\cdot m(J)=m(J).$$

In order to define the conjugacy map between f_p and σ, we first observe that associated to each $x\in\mathbb{R}/\mathbb{Z}$ there exists a unique sequence $(b_n)_{n\in\mathbb{N}}$ defined by its orbit via f_p, i.e., if we set $x_n=f_p^n(x)$, then we define the sequence $(b_n)_n$ as

$$b_n=\begin{cases}0 & \text{if } x_n\in I_0(p)\\ 1 & \text{if } x_n\in I_1(p)\end{cases}. \tag{6}$$

Since the points p and 1 are sent to the same sequence, it is indeed more convenient to work with $\mathbb{R}/\mathbb{Z}$ instead of $[0,1]$. So we define the conjugacy map $\pi_p\colon\mathbb{R}/\mathbb{Z}\to\Sigma$ by $\pi_p(x)=(b_n)_n$. This is a bijection that preserves measure, since for every measurable set $A\subset\Sigma$, we have $m(\pi_p^{-1}(A))=\mu_p(A)$. Indeed, since the cylinders $[i:0]=\{(b_n)_n\mid b_i=0\}$ and $[i:1]=\{(b_n)_n\mid b_i=1\}$ generates the σ-algebra of Σ and since we have

$$m(\pi_p^{-1}([i:0]))=m\left(f_p^{-i}([0,p)\right)=m([0,p))=p=\mu_p([i:0]),$$

and

$$m(\pi_p^{-1}([i:1]))=m\left(f_p^{-i}([p,1]\right)=m([p,1])=1-p=\mu_p([i:1]),$$

we conclude π_p preserves measure. $\square$

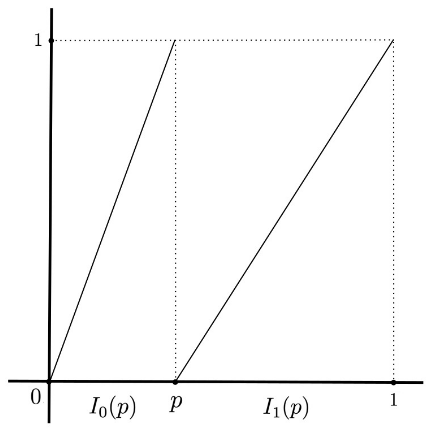

Figure 4: The graph of the function f_p.

3.1.1 Construction of the set

Now we construct the full measure set E we promised. Until now we were only dealing with the Lebesgue measure on $\mathbb{R}/\mathbb{Z}$ and calling it m. But since E is going to be a measurable set of $(0,1)\times\mathbb{R}/\mathbb{Z}$, we will call m the Lebesgue measure on $(0,1)\times\mathbb{R}/\mathbb{Z}$, m_1 the Lebesgue measure on $(0,1)$, and m_2 the Lebesgue measure on $\mathbb{R}/\mathbb{Z}$.

For a fixed $x\in\mathbb{R}/\mathbb{Z}$, consider the number of iterates $f_p^i(x)$ that enters the set $I_1(p)$. In an equivalent way, for a fixed $x\in\mathbb{R}/\mathbb{Z}$, consider the number of times the number 1 appears in the sequence $(b_n)_n$ associated with the orbit of x by f_p. By the Law of Large Numbers, the frequency of those $1's$ converges to the measure of the interval $I_1(p)$ for Lebesgue almost every $x\in\mathbb{R}/\mathbb{Z}$:

$$\lim_{n\to\infty}\frac{\#\{0\le i\le n-1\mid f_p^i(x)\in I_1(p)\}}{n}=1-p,$$

for m-every $x\in\mathbb{R}/\mathbb{Z}$.

Now, define the set $E\subset(0,1)\times\mathbb{R}/\mathbb{Z}$ by:

$$E=\left\{(p,x)\in(0,1)\times\mathbb{R}/\mathbb{Z}\mid \lim_{n\to\infty}\frac{\#\{0\le i\le n-1\mid f_p^i(x)\in I_1(p)\}}{n}=1-p\right\}. \tag{7}$$

Proposition 2. *The set E defined by (7) is a measurable subset of* $(0,1)\times\mathbb{R}/\mathbb{Z}$ *and has full measure, i.e.,* $m(E)=1$.

Proof. To see that E is measurable, define the map $F\colon(0,1)\times\mathbb{R}/\mathbb{Z}\to(0,1)\times\mathbb{R}/\mathbb{Z}$ by $F(p,x)=(p,f_p(x))$, and the consider the measurable set $A=\{(p,x)\in(0,1)\times\mathbb{R}/\mathbb{Z}\mid x\ge p\}$. Also define, for each $n\in\mathbb{N}$, a function $\beta_n\colon(0,1)\times\mathbb{R}/\mathbb{Z}\to\mathbb{R}$ by

$$\beta_n(p,x)=\#\{0\le i\le n-1\mid F^i(p,x)\in A\}.$$

This is a measurable function since F and A are, and $\beta_n=\chi_A+\chi_A\circ F+\cdots+\chi_A\circ F^{n-1}$. Now, one has just to observe that

$$E=\bigcap_{m=1}^{\infty}\bigcup_{N=1}^{\infty}\bigcap_{n=N}^{\infty}\beta_n^{-1}\left(n\left(1-p-m\right),n\left(1-p+m\right)\right),$$

so that E is indeed a measurable subset of $(0,1) \times \mathbb{R}/\mathbb{Z}$.

Since E is a measurable set, we can calculate its measure using Fubini's Theorem: if we call L_p the leaves of the foliation of $(0,1) \times \mathbb{R}/\mathbb{Z}$ by horizontal lines, i.e., each L_p is the horizontal circle $p \times \mathbb{R}/\mathbb{Z}$, then

$$m(E) = \int_{(0,1)} \int_{p \times L_p} \chi_E(p,x) dm_2(x) dm_1(p) = \int_0^1 m_2(E \cap L_p) dm_1(p) = 1,$$

since $m_2(E \cap L_p) = 1$. $\square$

3.1.2 Construction of the foliation

The idea is to construct a foliation $\mathcal{F}$ by smooth curves L_α in such a way that two points (p,x) and (q,y) belong to the same leaf L_α if and only if they have the same sequence expansion in Σ, i.e., $\pi_p(x) = \pi_q(y)$.

The precise construction goes as follows: given $\alpha \in \mathbb{R}/\mathbb{Z}$ let $a = (a_n)_n$ be its binary expansion, i.e., $a = \pi_{1/2}(\alpha)$ and $\alpha = \sum_{n=1}^{\infty} a_n/2^n$. Define

$$L_\alpha = \{(p,x) \in (0,1) \times \mathbb{R}/\mathbb{Z} \mid \pi_p(x) = a\},$$

for each $\alpha \in (0,1)$.

Claim 1. *The family $\{L_\alpha\}_\alpha$ forms a partition of $(0,1) \times \mathbb{R}/\mathbb{Z}$:*

- *if $\alpha \neq \beta$ then $L_\alpha \cap L_\beta = \emptyset$;*
- $(0,1) \times \mathbb{R}/\mathbb{Z} = \bigcup_\alpha L_\alpha$.

Proof. Consider $\alpha, \beta \in (0,1)$ and suppose $a = (a_n)_n$ and $b = (b_n)_n$ to be such that $a = \pi_{1/2}(\alpha)$ and $b = \pi_{1/2}(\beta)$. If we have a point $(p,x) \in L_\alpha \cap L_\beta$, then we would have $\pi_p(x) = a$ and $\pi_p(x) = b$, i.e., $a_i = b_i$ for every i. Since the binary expansion is unique, this implies $\alpha = \beta$. Therefore, if $\alpha \neq \beta$ we must have $L_\alpha \cap L_\beta = \emptyset$.

Next, we prove $(0,1) \times \mathbb{R}/\mathbb{Z} = \bigcup_\alpha L_\alpha$. Since the direction $(0,1) \times \mathbb{R}/\mathbb{Z} \supseteq \bigcup_\alpha L_\alpha$ is trivial, we prove the other one: let (p,x) be a point in $(0,1) \times \mathbb{R}/\mathbb{Z}$. We need to find an α such that $\pi_p(x) = a$. But is straightforward, since if we pick the sequence $\pi_p(x) = (\tilde{a}_n)_n$ of $0's$ and $1's$, there is an $\tilde{\alpha}$ such that its binary expansion is $(\tilde{a}_n)_n$, namely $\tilde{\alpha} = \sum_{n=1}^{\infty} \tilde{a}_n/2^n$. $\square$

Claim 2. *For each α there exists a real analytic function $\phi_\alpha \colon (0,1) \to \mathbb{R}$ such that L_α is the graph of ϕ_α, i.e.,*

$$L_\alpha = \{(p, \phi_\alpha(p)) \mid p \in (0,1)\}.$$

Proof. Fix a pair $(p,x) \in L_\alpha$ (i.e., $\pi_p(x) = a$). Call $\phi_\alpha(p) = x$ and define $(x_n)_n$ inductively by setting $x_1 = x$ and $x_{n+1} = f_p(x_n)$, so that $x_n \in I_{a_n}(p)$ for all $n \in \mathbb{N}$. Hence,

$$x_{n+1} = \begin{cases} \frac{x_n}{p} & \text{if } a_n = 0 \\ \frac{x_n - p}{1-p} & \text{if } a_n = 1 \end{cases}.$$

If we write, $p(0) = p$ and $p(1) = 1 - p$, the relation between x_{n+1} and x_n can be stated as:

$$x_{n+1} = \frac{x_n - a_n p(0)}{p(a_n)},$$

so that $x_n = a_n p(0) + p(a_n)x_{n+1}$.

This relation allow us to write $x = \phi_\alpha(p)$ in terms of the sequence $(a_n)_n$. Indeed,

$$\begin{aligned}
\phi_\alpha(p) = x = x_1 &= a_1 p(0) + p(a_1)x_2 \\
&= a_1 p(0) + p(a_1)\left(a_2 p(0) + p(a_2)x_3\right) \\
&= p(0)\left(a_1 + p(a_1)a_2\right) + p(a_1)p(a_2)x_3 \\
&= p(0)\left(a_1 + p(a_1)a_2 + p(a_1)p(a_2)a_3\right) + p(a_1)p(a_2)p(a_3)x_4,
\end{aligned}$$

and so on. If we define $\psi_n(p) = p(a_1)p(a_2)\cdots p(a_{n-1})$, we will obtain the following formula for $\phi_\alpha(p)$:

$$\phi_\alpha(p) = p(0)\sum_{n=1}^{\infty} \psi_n(p)a_n. \tag{8}$$

Observer that for each N, the function $\Psi_N = p(0)Sum_{n=1}^{N}\psi_n(p)a_n$ is analytic on p, and that for every small $\varepsilon > 0$, the sum converges uniformly on the interval $[\varepsilon, 1-\varepsilon]$ (indeed, for every $p \in [\varepsilon, 1-\varepsilon]$, we have $\psi_n(p) \leq (1-\varepsilon)^n$). Evermore, this convergence extends to complex values of p in the disk of center (1/2,0) and radius $1/2 - \varepsilon$ in the complex plane. Therefore, by Weierstrass' uniform convergence theorem, we conclude the function ϕ_α is analytic in p. □

Claim 3. *For each α, the leaf L_α intersects E in at most one point.*

Proof. Indeed, fix α and suppose (p, x) and (q, y) are both in $E \cap L_\alpha$. Since they are both in E, we must have:

$$\lim_{n\to\infty} \frac{\#\{0 \leq i \leq n-1 \mid f_p^i(x) \in I_1(p)\}}{n} = 1 - p,$$

and

$$\lim_{n\to\infty} \frac{\#\{0 \leq i \leq n-1 \mid f_q^i(y) \in I_1(q)\}}{n} = 1 - q.$$

On the other hand, we know that if $(k_n)_n = \pi_p(x)$ is the sequence associated to x by the orbit of f_p and $(\ell_n)_n = \pi_q(y)$ is the sequence associated to y by the orbit of f_q, then

$$1 - p = \lim_{n\to\infty} \frac{\#\{0 \leq i \leq n-1 \mid f_p^i(x) \in I_1(p)\}}{n} = \lim_{n\to\infty} \frac{\#\{0 \leq i \leq n-1 \mid k_i = 1\}}{n},$$

and

$$1 - q = \lim_{n\to\infty} \frac{\#\{0 \leq i \leq n-1 \mid f_q^i(y) \in I_1(q)\}}{n} = \lim_{n\to\infty} \frac{\#\{0 \leq i \leq n-1 \mid \ell_i = 1\}}{n}.$$

On the other hand, since (p, x) and (q, y) both belong to L_α, we must have $\pi_p(x) = \pi_q(y) = a$, where $a = (a_n)_n$ is the sequence $\pi_{1/2}(\alpha)$ as stated above. Then, $k_i = \ell_i$ for all $i \in \mathbb{N}$, and so

$$\lim_{n\to\infty} \frac{\#\{0 \leq i \leq n-1 \mid k_i = 1\}}{n} = \lim_{n\to\infty} \frac{\#\{0 \leq i \leq n-1 \mid \ell_i = 1\}}{n},$$

implying $1 - p = 1 - q$. Therefore, we must have $p = q$. Next, we use the fact that, fixed $p \in (0, 1)$, the map π_p is a bijection: since $p = q$ and $k_i = \ell_i$ for all i, we conclude that $\pi_p(x) = (k_n)_n = (\ell_n)_n = \pi_q(y)$ and also that $\pi_p(x) = \pi_q(y) = \pi_p(y)$, so $x = y$ and (p, x) and (q, y), as desired.[2] □

[2]Notice, however, that the sequence of $1's$ in α could not be defined (it is only defined almost everywhere). In this case, L_α not even intersects E.

3.1.3 How about the regularity of the foliation?

Until now, we have only proved assertions about the regularity of the leaves. However, what can we say about the regularity of the foliation itself?

At the beginning of Section 3 we explained that if the foliation is in some sense regular, the above weird example cannot occur. In fact, if the foliation is of regularity C^1, a foliated chart will send the Lebesgue measure to a measure that is absolutely continuous with respect to the Lebesgue measure (and we can show that things must behave well in the sense of these notes).

The problem with the foliation we have just constructed is that it is only continuous: one can prove that the map $(p, \alpha) \mapsto (p, \phi_\alpha(p))$ maps $(0, 1) \times \mathbb{R}/\mathbb{Z}$ homeomorphically onto itself.

In conclusion, we found a C^0-foliation $\mathcal{F}$ of $(0, 1) \times \mathbb{R}/\mathbb{Z}$ by analytic leaves, and a measurable set $E \subseteq (0, 1) \times \mathbb{R}/\mathbb{Z}$ such that each leaf of $\mathcal{F}$ intersects E in at most one point.

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